

Galois Unitaries, Mutually Unbiased Bases, and MUB-balanced states

D.M. Appleby

*Centre for Engineered Quantum Systems, School of Physics, The University of
Sydney, Sydney, NSW 2006, Australia*

Ingemar Bengtsson

Stockholms Universitet, AlbaNova, Fysikum, S-106 91 Stockholm, Sweden

Hoan Bui Dang

*Perimeter Institute for Theoretical Physics and University of Waterloo, Waterloo,
Ontario, Canada*

Abstract

A Galois unitary is a generalization of the notion of anti-unitary operators. They act only on those vectors in Hilbert space whose entries belong to some chosen number field. For Mutually Unbiased Bases the relevant number field is a cyclotomic field. By including Galois unitaries we are able to remove a mismatch between the finite projective group acting on the bases on the one hand, and the set of those permutations of the bases that can be implemented as transformations in Hilbert space on the other hand. In particular we show that there exist transformations that cycle through all the bases in all dimensions $d = p^n$ where p is an odd prime and the exponent n is odd. (For even primes unitary MUB-cyclers exist.) These transformations have eigenvectors, which are MUB-balanced states (i.e. rotationally symmetric states in the original terminology of Wootters and Sussman) if and only if $d = 3$ modulo 4. We conjecture that this construction yields all such states in odd prime power dimension.

1. INTRODUCTION

Symmetries in quantum mechanics are described by unitary or anti-unitary operators [1]. Recently it was noted that the notion of anti-unitary operators can be generalized to something dubbed “g-unitaries”, where the “g” stands for Galois and means that these operations can be applied only to suitably restricted vectors in Hilbert space [2]. This new idea deserves to be followed up. Here we will show that g-unitaries have a role to play when thinking about a problem raised by Arnold in his last book [3]. His starting point is the observation that certain finite sets can be interpreted as projective lines over finite Galois fields (and we add that the name of Galois now occurs for the second time for a separate reason). One of Arnold’s problems is to understand the intrinsic characterization of those permutations of the set that arise as projective permutations. He also asks for applications to the natural sciences. We will argue that in fact this problem is relevant when one is trying to understand the shape of quantum state spaces, and that g-unitaries enter at a key point of the argument.

What we have in mind is the well known fact that Mutually Unbiased Bases (MUB) form a projective line over a finite Galois field, when the dimension d of Hilbert space is a power of a prime number p . This is connected to the existence of the finite affine plane on which discrete Wigner functions are defined [4]. There are $d + 1$ MUB and altogether $d(d + 1)$ unit vectors involved. It turns out that the permutations of bases that can be effected by unitary transformations are in fact the projective transformations that interested Arnold. However, if d is odd there are additional projective transformations that do not arise in this way. It is at this point that g-unitaries enter the story to provide some of the missing symmetries. Moreover they have a geometric interpretation in terms of rotations in Bloch space. When viewed as projection operators the MUB vectors give rise to a polytope which can be described by noting that the bases span $d + 1$ totally orthogonal planes in Bloch space [5]. It plays a major role in fault tolerant quantum computation [6], and in prime dimensions it is known as the stabilizer polytope. To avoid a possible ambiguity in prime power dimensions we will instead refer to it as the complementarity polytope. (When the dimension is a power of a prime the relevant Heisenberg group yields a set of several interlocking MUBs. We focus on just one of them.) The symmetry group of the complementarity polytope gives rise to arbitrary permutations of the $d + 1$ planes spanned by the bases, but there is a natural restriction which gives rise to precisely the projective permutations including those effected by g-unitaries.

We will pay special attention to MUB-cyclers, that is transformations of order $d + 1$ cycling through the entire set of MUB. If d is even unitary MUB cyclers exist [7, 8]. If $d = p^n$ is odd and $d = 3 \bmod 4$, anti-unitary MUB-cyclers exist [9]. We will find g-unitary MUB-cyclers for all $d = p^n$ where n is odd. This seems to us to be an interesting fact, even though we freely admit that unitary MUB-cyclers are the important ones from the point of view of the experimentalist: they enable us to reach any MU basis by iterating a single operation in the lab. Implementing g-unitaries in the lab will be very hard! However, they remain interesting from the point of view of Arnold’s problem.

One important reason why MUB-cyclers deserve attention is that one expects vectors invariant under a MUB-cycler to be MUB-balanced states in the terminology of Amburg et al [10] (or rotationally symmetric states in the original terminology of Wootters and Sussman [7]). Such states are defined by the property that the $d + 1$ probability vectors obtained by projection to the $d + 1$ MUB are identical, up to permutations of the components. MUB-balanced states were originally constructed in even prime power dimension by Wootters and Sussman [7], and Amburg et al [10] recently gave a construction for odd prime power dimensions equal to 3 modulo 4. Amburg et al took notice of some interesting properties possessed by these states, and expressed their surprise that such states exist at all. Indeed, according to these authors, MUB-balanced states are interesting because they “have no right to exist”. The same can be said for the MUB themselves, and also for the SICs that we will mention below.

The existence of MUB cycling g-unitaries does not settle the existence of MUB-balanced states, since—unlike unitaries—such operators need not have any eigenvectors at all. We will establish that the ones we are considering do. At this point we fully expected that we would obtain a large supply of MUB-balanced states, but further analysis shows that a vector invariant under a MUB cycling g-unitary is in fact a MUB-balanced state if and only if it is also left invariant by a MUB-cycling anti-unitary. The latter exist only in prime power dimensions $d = 3 \bmod 4$. In these dimensions we are able to prove that the MUB-balanced states arising from MUB cyclers belong to a single orbit under the Clifford group. The obstruction that arises when $d = 1 \bmod 4$ has to do with the fact that g-unitaries do not preserve Hilbert space norm in general [2].

The states constructed by Amburg et al. [10] are representatives of the single orbit that we found. We conjecture that these authors have in fact found all MUB-balanced states up to the action of the Clifford group. This highlights the remarkable position of these states within quantum state space.

MUB-balanced states belong to the wider class of Minimum Uncertainty States (MUS) [7]. Geometrically a state is a MUS if the probability vectors arising from projections to the MUB have equal length. Simple parameter counting suggests that the set of MUS in a Hilbert space of complex dimension d has $d - 2$ real dimensions. While not very exceptional in themselves, the set of MUS does include all the MUB-balanced states, and surprisingly—provided the dimension is a prime number—they also include all Weyl-Heisenberg covariant SICs [11]. Following these observations Wootters and Sussman and (independently of them) one of the present authors (DMA) obtained some results concerning MUB-balanced states in odd prime power dimensions. However these results were not published in a journal at the time (though some of them did appear in ref. [12]). There was then a lapse of six years after which, quite independently, both the present authors and Wootters and Sussman [10] (together with Amburg and Sharma) decided to return to the problem. The approach taken in ref. [10] is very different from ours, and we believe that our approach provides a useful complement.

We have saved a major motivation for this work for last. The whole discussion will rest on the fact that, in the natural basis singled out by the group, all the entries of the MUB vectors belong to the cyclotomic field generated by the roots of unity. The cyclotomic field is an abelian extension of the rational field \mathbb{Q} . This may seem like an overly sophisticated way of expressing the fact that the entries

are roots of unity, but it should be remembered that there is every reason to believe that the entries of the vectors forming what is known as a SIC-POVM belong to an abelian extension of the real quadratic field $\mathbb{Q}(\sqrt{(d-3)(d+1)})$ [2]. The latter is in itself an extension over the rationals, but its abelian extensions are of a rather mysterious kind. In fact they form the subject of Hilbert's 12th problem [13]. The SICs themselves are very distinguished orbits of the Weyl-Heisenberg group, and they are prime examples of sets of states that “have no right to exist”. Still, it seems that they do, at least in low dimensions. Besides their own intrinsic interest they are important technically, due to their applications to quantum tomography, quantum cryptography and entanglement detection (see, for example, refs. [14, 15, 16]), and also conceptually, due to their role in the qbist program (see, for example, ref. [17]). So their properties and significance deserve to be better understood. At the moment, their existence in arbitrary dimensions remains a tantalizing conjecture only [18, 19, 20, 21]. The notion of g-unitaries arose in an attempt to understand them better [2], and from this point of view we are investigating a toy model for SICs.

2. CHOICE OF DIMENSIONS AND ORGANIZATION OF OUR PAPER

The story that we have to tell depends sensitively on the dimension d of Hilbert space, and becomes especially transparent if d is an odd prime number p . The case when $d = p^n$ is a power of an odd prime number is partly a straightforward generalization but does require rather more in the way of notation, and moreover the argument diverges from the $d = p$ case at some key points.

Section 3 introduces the minimal amount of background concerning fields and Galois extensions that we will need in this paper. We then introduce the Clifford group and its extension to g-unitaries in sections 4 and 5. In order to increase readability we confine these two sections to the case when the dimension is an odd prime number, and discuss the general case $d = p^n$ (with p odd) in section 6. In section 7 we introduce the complementarity polytope, and in section 8 we give a geometric interpretation of the g-unitaries in terms of its symmetry group. In section 9 we prove a result concerning MUB-cyclers. There is a significant difference depending on whether the exponent n , where $d = p^n$, is odd or even. In section 10 we prove that g-unitary MUB-cyclers do have eigenvectors, and in section 11 we establish that these eigenvectors are MUB-balanced states when $d \equiv 3 \pmod{4}$. Since the story we tell becomes involved and (we are afraid) makes some demands on the reader's time, we have tried to summarize our main results in words in section 12. The results are in fact simple and (we think) appealing.

3. FIELDS AND FIELD EXTENSIONS

This review section is intended to be a brief introduction to fields, field extensions, Galois automorphisms, and finite fields, for readers with little or no background in field theory. The role of finite fields in the theory of Mutually Unbiased Bases was made clear early on [22], but here we will need an infinite number field as well. Since the main purpose is to help our readers quickly grasp the key concepts used in the paper, we avoid unnecessarily technical definitions and derivations as much as we can. Rigorous treatments of the subjects can be found in textbooks on fields and Galois theory [23, 24, 25].

Fields. A field \mathbb{F} is a set with two commutative operations, addition and multiplication, that are compatible via distributivity. Furthermore, \mathbb{F} has an additive identity 0, a multiplicative identity 1, and every element in \mathbb{F} has an additive inverse and, with the exception of 0, a multiplicative inverse. Thus \mathbb{F} is a group under addition, and \mathbb{F} with the 0 element removed is a group under multiplication. Common examples include the field of rational numbers \mathbb{Q} and the field of real numbers \mathbb{R} , with the usual addition and multiplication. There also exist finite fields, i.e. fields with a finite number of elements. For example, if p is a prime number, then the set of integers $\mathbb{Z}_p = \{0, 1, \dots, p-1\}$ with addition and multiplication modulo p forms a field, called a prime field.

Field extensions. The complex field \mathbb{C} commonly used in quantum physics is constructed from the real field \mathbb{R} by adding to it an imaginary number i defined by the property $i^2 = -1$, in other words i is defined to be a root of the real polynomial $x^2 + 1$. The complex field is then defined as the set of all numbers of the form

$$\mathbb{C} = \{a + ib : a, b \in \mathbb{R}\} \quad (1)$$

where the sum and product of any two elements $(a_1 + ib_1)$ and $(a_2 + ib_2)$, using the identity $i^2 + 1 = 0$, can be easily worked out to be $(a_1 + a_2) + i(b_1 + b_2)$ and $(a_1a_2 - b_1b_2) + i(a_1b_2 + a_2b_1)$, which are clearly also in \mathbb{C} . One can think of a complex number $(a + ib)$ as a 2-component vector (a, b) in a real vector space. The dimensionality of this vector space is equal to the degree of the polynomial defining i , namely 2.

More generally, given a field \mathbb{F} and a number $h \notin \mathbb{F}$, we can construct a field \mathbb{E} such that it is the smallest field containing both h and \mathbb{F} , denoted as $\mathbb{E} \equiv \mathbb{F}(h)$. \mathbb{F} is called the ground field, \mathbb{E} is called the extended field, and the field extension is denoted as \mathbb{E}/\mathbb{F} (reads as \mathbb{E} over \mathbb{F}). We assume that h is algebraic over \mathbb{F} , i.e. it is a root of some polynomial with coefficients in \mathbb{F} . Among polynomials that admits h as a root, consider one with the lowest degree, and let n denote its degree. The extended field \mathbb{E} can then be constructed as

$$\mathbb{E} = \{f_0 + f_1h + \dots + f_{n-1}h^{n-1} : f_i \in \mathbb{F}\} . \quad (2)$$

\mathbb{E} can be thought of as an n -dimensional vector space over \mathbb{F} , and n is called the degree of the field extension \mathbb{E}/\mathbb{F} .

Galois automorphisms. Given a field extension \mathbb{E}/\mathbb{F} (\mathbb{E} is an extension of \mathbb{F}), a Galois automorphism of \mathbb{E}/\mathbb{F} is defined as an automorphism of \mathbb{E} that fixes the elements in \mathbb{F} . In other words, it is a bijective mapping $g : \mathbb{E} \rightarrow \mathbb{E}$ that satisfies:

- (1) $g(e_1 + e_2) = g(e_1) + g(e_2)$ for all $e_1, e_2 \in \mathbb{E}$
- (2) $g(e_1e_2) = g(e_1)g(e_2)$ for all $e_1, e_2 \in \mathbb{E}$
- (3) $g(f) = f$ for all $f \in \mathbb{F}$

All such automorphisms form a group called the Galois group of the extension \mathbb{E}/\mathbb{F} , denoted by $\text{Gal}(\mathbb{E}/\mathbb{F})$. The order of the Galois group is less than or equal to the degree of the extension, and when they are equal we call such a field extension a Galois extension. All the field extensions considered in this paper are Galois extensions.

Let us go back to the example of the extension of the real field \mathbb{R} to the complex field \mathbb{C} . If $g : \mathbb{C} \rightarrow \mathbb{C}$ is a Galois automorphism of the extension \mathbb{C}/\mathbb{R} , then it satisfies

$$g(i)g(i) = g(i^2) = g(-1) = -1 , \quad (3)$$

which implies either $g(i) = i$ or $g(i) = -i$. Note that the value of $g(i)$ completely specifies the Galois automorphism g because its action on any complex number $(a + ib)$ can be expressed as $g(a + ib) = g(a) + g(i)g(b) = a + g(i)b$. In this case the two Galois automorphisms in the Galois group corresponding to $g(i) = i$ and $g(i) = -i$ are the identity mapping and complex conjugation. This extension is a Galois extension, as the group has order 2, equal to the degree of the extension.

Cyclotomic fields. A cyclotomic field is generated by extending the rational field \mathbb{Q} with an N -th root of unity $\omega = e^{2\pi i/N}$. Although cyclotomic fields can be defined for any N , we will restrict ourselves to the case when $N = p$ is a prime number. Then the minimal polynomial of ω over \mathbb{Q} is

$$P(x) = 1 + x + \dots + x^{p-1} . \quad (4)$$

The cyclotomic field $\mathbb{Q}(\omega)$ is then defined by

$$\mathbb{Q}(\omega) = \{q_0 + q_1\omega + \dots + q_{p-2}\omega^{p-2} : q_i \in \mathbb{Q}\} . \quad (5)$$

The extension $\mathbb{Q}(\omega)/\mathbb{Q}$ is of degree $p - 1$. Let $g : \mathbb{Q}(\omega) \rightarrow \mathbb{Q}(\omega)$ be a Galois automorphism of the field extension. Just like in the complex case, g is completely specified by the value of $g(\omega)$. The identity

$$(g(\omega))^p = g(\omega^p) = g(1) = 1 , \quad (6)$$

implies that $g(\omega) = \omega^k$ for some $1 \leq k \leq p - 1$. If we specifically denote g_k to be the Galois automorphism that maps ω into ω^k , then $\{g_k\}_{k=1}^{p-1}$ is the order $p - 1$ Galois group of this field extension. One can see that g_1 is the identity mapping, and g_{p-1} is complex conjugation.

Finite fields. A finite field (somewhat confusingly, also called a Galois field) is a field that has a finite number of elements, called its order. The prime field \mathbb{Z}_p for any prime number p is an example of a finite field, as previously mentioned. There are also finite fields of other orders. However, it is known that finite fields only exist for which the order is a prime power p^n , and for every prime power there exists a unique (up to isomorphism) field of this order. Thus, we can refer to a finite field only by its order, and we shall denote the finite field of order N by \mathbb{F}_N , where N must be a prime power for \mathbb{F}_N to exist. We will not provide the proof here, but will instead give a concrete example of how to generate \mathbb{F}_{p^n} by extending the prime field \mathbb{F}_p .

Consider the finite field $\mathbb{F}_3 = \{0, 1, 2\}$. If we take the square (modulo 3) of each element in \mathbb{F}_3 we will get the set $\{0, 1\}$, which, if we exclude 0, is called the set of quadratic residues, i.e. the set of non-zero elements that can be expressed as a square of some element in the field. 2 is a quadratic non-residue because no element in \mathbb{F}_3 squares to 2, therefore the polynomial $x^2 + 1 \equiv x^2 - 2 \pmod{3}$ has no root in \mathbb{F}_3 and is irreducible over \mathbb{F}_3 . If we define λ to be the root of $x^2 + 1$, a finite field version of the imaginary number i , then we can adjoin λ to \mathbb{F}_3 to create the extended field

$$\mathbb{F}_3(\lambda) = \{a + \lambda b : a, b \in \mathbb{F}_3\} . \quad (7)$$

One can easily see that $\mathbb{F}_3(\lambda) \equiv \mathbb{F}_9$ has 9 elements and that its multiplication table can be calculated using the identity $\lambda^2 + 1 = 0$. In general, if we start from an irreducible polynomial of degree r in \mathbb{F}_p , we will be able to extend the field to \mathbb{F}_{p^r} .

There are a few basic properties of finite fields relevant to our paper that we would like to mention here. First of all every finite field admits one (and in fact several) primitive element θ , that is an element such that every non-zero element x in the field can be written as $x = \theta^r$ for some choice of the exponent r . Then, for \mathbb{F}_p and all extension fields \mathbb{F}_{p^n} , the following facts hold:

- (1) $a^{p^n} = a \quad \forall a \in \mathbb{F}_{p^n}$
- (2) $(a + b)^p = a^p + b^p \quad \forall a, b \in \mathbb{F}_{p^n}$
- (3) $\forall a \in \mathbb{F}_{p^n}, a^p = a$ if and only if $a \in \mathbb{F}_p$.

4. THE CLIFFORD GROUP AND MUTUALLY UNBIASED BASES

This section is again a review of things that are fully described elsewhere [20, 9]. We begin in odd prime dimensions d , where the Weyl-Heisenberg group has an essentially unique representation given by a primitive root of unity and by Sylvester's clock and shift matrices

$$\omega = e^{\frac{2\pi i}{d}}, \quad Z|x\rangle = \omega|x\rangle, \quad X|x\rangle = |x+1\rangle. \quad (8)$$

The labels on the states are integers modulo the dimension d , and in this and in the following section d is taken to be an odd prime. The group elements are best organized into the displacement operators

$$D_{\mathbf{u}} = \omega^{\frac{u_1 u_2}{2}} X^{u_1} Z^{u_2}, \quad (9)$$

where \mathbf{u} is a two component vector with entries u_1, u_2 chosen from the integers modulo d . The latter form the finite field \mathbb{Z}_d , and $1/2$ denotes the multiplicative inverse of 2 in this field. The reason for inserting the phase factors into the definition of the displacement operators is that the group law then takes the useful form

$$D_{\mathbf{u}} D_{\mathbf{v}} = \omega^{\Omega(\mathbf{u}, \mathbf{v})} D_{\mathbf{u}+\mathbf{v}}, \quad \Omega(\mathbf{u}, \mathbf{v}) = u_2 v_1 - u_1 v_2. \quad (10)$$

Here Ω is a symplectic form on the vector space $(\mathbb{Z}_d)^2$.

The Clifford group is defined as the normalizer of the Weyl-Heisenberg group within the unitary group. Modulo phases it is isomorphic to a semi-direct product of the special linear group $\text{SL}(2, \mathbb{Z}_d)$ (which is isomorphic to the symplectic group) acting on the discrete translation group $(\mathbb{Z}_d)^2$. This action is

$$U_G D_{\mathbf{u}} U_G^{-1} = D_{G\mathbf{u}}. \quad (11)$$

It is then obvious from the group law (10) that the matrices G have to preserve the symplectic form—i.e. they must have determinant unity, and belong to $\text{SL}(2, \mathbb{Z}_d)$. Their unitary representation U_G is fixed up to overall phase factors. In this paper we will need the latter too, so we use the metaplectic representation which is faithful as opposed to only projective. It is given by [9]

$$G = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \rightarrow \begin{cases} U_G = \frac{e^{i\theta}}{\sqrt{d}} \sum_{r,s} \omega^{\frac{1}{2\beta}(\delta r^2 - 2rs + \alpha s^2)} |r\rangle\langle s| & \beta \neq 0 \\ U_G = l(\alpha) \sum_s \omega^{\frac{\alpha\gamma}{2}s^2} |\alpha s\rangle\langle s| & \beta = 0 \end{cases} \quad (12)$$

where $\alpha\delta - \beta\gamma = 1$ modulo d and

$$e^{i\theta} = -\frac{1}{i^{\frac{d+3}{2}}} l(-\beta) = \begin{cases} (-1)^k l(-\beta) & \text{if } d = 4k + 1 \\ (-1)^{k+1} i l(-\beta) & \text{if } d = 4k + 3 \end{cases} \quad (13)$$

$$l(x) = \begin{cases} 1 & \text{if } x \in \mathbf{Q} \\ -1 & \text{if } x \in \mathbf{N} . \end{cases} \quad (14)$$

Number theorists know $l(x)$ as the Legendre symbol. \mathbf{Q} is the set of quadratic residues, that is to say the set of non-zero integers modulo d that can be written as the square of another integer modulo d , and \mathbf{N} is the set of quadratic non-residues. These two sets have the same size.

With the Clifford group in hand (and d still set firmly equal to an odd prime) we can obtain a complete set of $d + 1$ MUB in Hilbert space. We simply choose a vector belonging to the computational basis, and act on it with the Clifford group. The resulting orbit consists of $d(d + 1)$ unit vectors forming $d + 1$ MUB. We denote these vectors by

$$|e_r^{(z)}\rangle , \quad z \in \{0, 1, \dots, d-1, \infty\} , \quad r \in \{0, 1, \dots, d-1\} . \quad (15)$$

where z labels the bases and r labels the vectors in a basis. Thus

$$|\langle e_r^{(z)} | e_{r'}^{(z')} \rangle|^2 = \frac{1}{d} + \delta_{z,z'} \left(\delta_{r,r'} - \frac{1}{d} \right) . \quad (16)$$

This amazing fact is by now sufficiently well known [26, 22, 27], so let us just mention that each individual basis is an eigenbasis of a cyclic subgroup of the Weyl-Heisenberg group. In general the Weyl-Heisenberg group permutes the vectors within the bases, while a symplectic unitary transforms the bases among themselves. To be precise, given a symplectic matrix of the form given in eq. (12) the bases are permuted by the Möbius transformation [28]

$$z \rightarrow z' = \frac{\alpha z + \beta}{\gamma z + \delta} . \quad (17)$$

This can be interpreted as a projective transformation of a finite projective line whose points are labelled by z , and it explains why one of the bases was labelled by ∞ . In particular

$$z' = \begin{cases} \infty & \text{if } z \neq \infty, \gamma z + \delta = 0 \text{ or } z = \infty, \gamma = 0 \\ \frac{\alpha}{\gamma} & \text{if } z = \infty, \gamma \neq 0 . \end{cases} \quad (18)$$

The individual vectors are also permuted by the Clifford group, up to phase factors that belong to the cyclotomic field.

There is an oddity here, since the Clifford group does not give us the most general Möbius transformation. The latter is of the form (17), with $\alpha\delta - \beta\gamma \neq 0$ but otherwise unrestricted. Such transformations are obtained from the general linear group $\text{GL}(2, \mathbb{Z}_d)$, consisting of all invertible two by two matrices with entries in \mathbb{Z}_d , by taking the quotient with all diagonal matrices. They form the projective group $\text{PGL}(2, \mathbb{Z}_d)$. Similarly, the special linear group $\text{SL}(2, \mathbb{Z}_d)$ gives rise to the projective group $\text{PSL}(2, \mathbb{Z}_d)$, which is a proper subgroup of PGL . In fact the respective orders of these groups are

$$|\mathrm{GL}| = (d-1)d(d^2-1) , \quad |\mathrm{PGL}| = d(d^2-1), \quad (19)$$

$$|\mathrm{SL}| = d(d^2-1) , \quad |\mathrm{PSL}| = \frac{d(d^2-1)}{2} . \quad (20)$$

Those elements in GL that do not belong to SL are easily identified. We observe that

$$\begin{pmatrix} 1 & 0 \\ 0 & x^2 \end{pmatrix} = \begin{pmatrix} x & 0 \\ 0 & x \end{pmatrix} \begin{pmatrix} x^{-1} & 0 \\ 0 & x \end{pmatrix} . \quad (21)$$

Elements of GL for which the determinant is a quadratic residue do not add anything to PGL beyond the contribution of SL. But elements whose determinant is not a quadratic residue do.

For a prime d , it is a number theoretical fact that -1 is a quadratic residue modulo d if and only if $d \equiv 1 \pmod{4}$. Matrices G having determinant -1 can be represented as acting on Hilbert space through anti-unitary transformations [20], and if $d \equiv 3 \pmod{4}$ the Clifford group extended to include such elements will yield general projective permutations of the bases [9]. But if $d \equiv 1 \pmod{4}$ every other projective permutation is missing. It is at this point that g-unitaries enter the story.

5. THE CLIFFORD GROUP EXTENDED BY g-UNITARIES

When we perform general linear transformations G in the discrete phase space symplectic areas will change according to $\Omega(\mathbf{u}, \mathbf{v}) \rightarrow \Delta \Omega(\mathbf{u}, \mathbf{v})$, where Δ is the determinant of the matrix G . To stay consistent with the group law (10) such a transformation must be accompanied with the Galois automorphism $\omega \rightarrow \omega^\Delta$. If not it will not be an automorphism of the Weyl-Heisenberg group, as we insist it should be. The question is whether such transformations are allowed.

Consider first the case $\det G = \Delta = -1$. In this case we are simply dealing with complex conjugation, and more generally with anti-unitary transformations of Hilbert space. This is certainly allowed, and takes us to the extended Clifford group, which is well understood [20]. As we noted in the previous section it will give rise to general projective permutations of the bases if $d \equiv 3 \pmod{4}$.

Now consider the case of general $\mathrm{GL}(2, \mathbb{Z}_p)$ matrices. Clearly

$$\det G = \Delta \quad \Rightarrow \quad G = \bar{G} K_\Delta , \quad \det \bar{G} = 1 , \quad K_\Delta = \begin{pmatrix} 1 & 0 \\ 0 & \Delta \end{pmatrix} . \quad (22)$$

So it will be enough to have a representation of the matrix K_Δ . We decide that

$$K_x = \begin{pmatrix} 1 & 0 \\ 0 & x \end{pmatrix} \quad (23)$$

is represented by the automorphism g_x ,

$$g_x : \quad \omega \rightarrow \omega^x . \quad (24)$$

This means that

$$U_G |\psi\rangle = U_{\bar{G}} g_\Delta (|\psi\rangle) . \quad (25)$$

The notation $g_x(|\psi\rangle)$ will always mean the vector obtained by applying the automorphism g_x to the components of $|\psi\rangle$ in the standard basis. The notation $g_x(A)$ is defined similarly using the matrix elements of the operator A . This action is defined only on matrices and vectors whose entries belong to the cyclotomic field.

Going back to the explicit expressions in eqs. (12-13) we see that there is a question whether the overall factor $e^{i\theta}/\sqrt{d}$ is in the cyclotomic field. To answer it we recall the Gaussian sum

$$\sum_{x=0}^{p-1} \omega^{x^2} = \begin{cases} \sqrt{p} & \text{if } p = 4k + 1 \\ i\sqrt{p} & \text{if } p = 4k + 3 . \end{cases} \quad (26)$$

We can rewrite this as

$$\sum_{x \in \mathbf{Q}} \omega^x - \sum_{x \in \mathbf{N}} \omega^x = \begin{cases} \sqrt{p} & \text{if } p = 4k + 1 \\ i\sqrt{p} & \text{if } p = 4k + 3 . \end{cases} \quad (27)$$

From this we reach the conclusion that the entries in the unitary matrices representing the symplectic group do indeed belong to the cyclotomic field, and then this is obviously true for the entire Clifford group. This means that we can use the Galois automorphisms of the cyclotomic field to represent the group $GL(2, \mathbb{Z}_d)$.

We must show that the representation is faithful. For this purpose write $G = \bar{G}K$, $\det \bar{G} = 1$, and consider

$$G_1 G_2 = \bar{G}_1 K_1 \bar{G}_2 K_2 = \bar{G}_1 (K_1 \bar{G}_2 K_1^{-1}) K_1 K_2 , \quad (28)$$

$$\bar{G}_2 = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \Rightarrow K_1 \bar{G}_2 K_1^{-1} = \begin{pmatrix} \alpha & \beta x_1^{-1} \\ \gamma x_1 & \delta \end{pmatrix} . \quad (29)$$

On the other hand

$$U_{G_1} U_{G_2} = U_{\bar{G}_1} g_1 (U_{\bar{G}_2} g_2) = U_{\bar{G}_1} g_1 (U_{\bar{G}_2}) g_1 g_2 . \quad (30)$$

Now it follows from eq. (27) that $g_1(e^{i\theta} \sqrt{p}) = e^{i\theta} \sqrt{p}$ if x_1 is a quadratic residue, and $g_1(e^{i\theta} \sqrt{p}) = -e^{i\theta} \sqrt{p}$ if it is not. In the latter case $(-\beta x_1^{-1}|p) = -(-\beta|p)$. It then follows by inspection of eqs. (12) that

$$g_1(U_{\bar{G}_2}) = U_{K_1 \bar{G}_2 K_1^{-1}} . \quad (31)$$

Given that the representation of SL is faithful [29, 9], the representation of GL is faithful too.

The sense in which the operators we are dealing with are g-unitary, as opposed to merely g-linear, was spelt out in ref. [2]. Suppose $G = \bar{G}K_\Delta$, where $\det \bar{G} = 1$. The adjoint of U_G is then defined by

$$\langle U_G \psi, \phi \rangle = g_\Delta^{-1}(\langle \psi, U_G^\dagger \phi \rangle) . \quad (32)$$

An explicit expression for the adjoint is

$$U_G^\dagger = g_\Delta^{-1} (U_{\bar{G}}^\dagger) g_\Delta^{-1} . \quad (33)$$

This expression is readily seen to imply that $U_G^\dagger U_G = U_G U_G^\dagger = \mathbf{1}$.

The action of the group is restricted to those vectors in Hilbert space whose components belong to the cyclotomic field $\mathbb{Q}(\omega)$. While this is a severe restriction, it does include the vectors in the MUB, since they form an orbit under the Clifford group. It also includes every vector that can be reached from the computational basis by using a larger set of transformations known as the Clifford hierarchy. This set is large enough for the purposes of universal quantum computation [30].

The g-unitaries will preserve the Hilbert space norm only if this norm is rational, which it may well not be. This means that their action is wildly discontinuous in general. Thus, consider the transformation $\omega \rightarrow \omega^2$ for $d = 5$, and its action on the vector

$$\begin{pmatrix} \omega^2 + \omega^3 \\ 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} \rightarrow \begin{pmatrix} \omega^4 + \omega \\ 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}. \quad (34)$$

These vectors are real, and can be approximated by rational vectors which are left invariant by the g-unitary, while the vector which is approximated is moved a long distance in Hilbert space. This behaviour should be kept in mind.

A g-unitary operator does preserve a norm which is obtained by multiplying the scalar product of two vectors with all its $d - 2$ Galois conjugates [31]. However, if this norm has a physical meaning it is hidden from us.

6. THE G-EXTENDED CLIFFORD GROUP IN THE PRIME POWER CASE

We hope that the idea of g-unitaries is by now clear, and turn to the complications that occur in prime power dimensions. We will assume the material in section 3.

When $d = p^n$ the elements of the Heisenberg group can be labelled by elements of the finite field \mathbb{F}_d of order p^n . Thus we write [9]

$$\omega = e^{\frac{2\pi i}{d}}, \quad Z_u|x\rangle = \omega^{\text{tr}(xu)}|x\rangle, \quad X_u|x\rangle = |x + u\rangle, \quad (35)$$

with $x, u \in \mathbb{F}_d$ and the field theoretic trace $\text{tr } u$ of an element of the finite field lies in the ground field \mathbb{F}_p . Although this is not immediately obvious the resulting group is isomorphic to the direct product of n copies of the Weyl-Heisenberg group in dimension p [32, 9]. The displacement operators are

$$D_{\mathbf{u}}|x\rangle = \omega^{\frac{1}{2} \text{tr}(u_1 u_2) + \text{tr}(u_2 x)}|x + u_1\rangle, \quad (36)$$

where the field theoretic trace appears again. Complete sets of MUB again exist [22]. They arise as eigenbases of maximally abelian subgroups with only the unit element in common just as they do in prime dimensions [27], but now there are many options for how to do this.

Two different Clifford groups can be defined [33]. The one we are interested in here is a subgroup of the other, and it has been called the restricted Clifford group [9]. It leaves a given complete set of MUB invariant, and includes symplectic unitaries representing the $\text{SL}(2, \mathbb{F}_d)$ matrices

$$G = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}. \quad (37)$$

A faithful representation is given by [9]

$$U_G = \begin{cases} l(\alpha) \sum_{x \in \mathbb{F}_d} \omega^{\text{tr}(\frac{\alpha\gamma}{2}x^2)} |\alpha x\rangle \langle x| & \text{if } \beta = 0 \\ \frac{\tilde{l}(-\beta)}{\sqrt{d}} \sum_{x,y \in \mathbb{F}_d} \omega^{\text{tr}(\frac{1}{2\beta}(\alpha y^2 - 2xy + \delta x^2))} |x\rangle \langle y| & \text{if } \beta \neq 0 \end{cases} \quad (38)$$

where

$$\tilde{l}(x) = \begin{cases} -i^{-\frac{n(p+3)}{2}} l(x) & x \neq 0 \\ 1 & x = 0 \end{cases} \quad (39)$$

and $l(x)$ is the quadratic character of \mathbb{F}_d , equal to $+1$ if x can be written as a square, and equal to -1 otherwise.

By adjoining the matrix K_x , given in eq. (23), we can extend the representation to include the group $\text{GL}_p(2, \mathbb{F}_d)$ consisting of two-by-two matrices whose determinants are non-zero and are in the ground field \mathbb{F}_p . For any $x \in \mathbb{F}_d$ the matrix K_x is represented in the same way as in section 5. Hence the case of prime power dimension differs from the prime dimensional case in that the g-extended Clifford group includes only a proper subgroup of GL.

We know that the representation of SL is faithful [29, 9], and given that we can prove that the representation of GL is faithful too:

Theorem 1. *The group $\text{GL}_p(2, \mathbb{F}_d)$ is faithfully represented by eqs. (38) provided that $d = p^n$ and n is odd.*

Proof. First recall the basic fact that $\mathbb{F}_p = \{x \in \mathbb{F}_d : x^p = x\}$. Let θ be a primitive element of \mathbb{F}_d . Then it is not difficult to show that $\theta_p = \theta^{1+p+\dots+p^{n-1}}$ belongs to \mathbb{F}_p , and is in fact a primitive element of \mathbb{F}_p (since this is true for all non-zero elements of \mathbb{F}_p).

Let G_1, G_2 be arbitrary elements of $\text{GL}_p(2, \mathbb{F}_d)$. We write $G_{1,2} = \bar{G}_{1,2} K_{\Delta_{1,2}}$, where $G_{1,2} \in \text{SL}(2, \mathbb{F}_d)$. Then

$$U_{G_1} U_{G_2} = U_{\bar{G}_1} g_{\Delta_1} (U_{\bar{G}_2}) g_{\Delta_2} . \quad (40)$$

We know that $\Delta_1 = \theta_p^u$ for some integer u . It is easily seen that $g_{\Delta_1} = g_{\theta_p^u}$, and that Δ_1 is a quadratic residue in \mathbb{F}_p if and only if u is even. Applying eq. (38) to

$$\bar{G}_2 = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \quad (41)$$

we have

$$U_{\bar{G}_2} = \begin{cases} l(\alpha) \sum_{x \in \mathbb{F}_d} \omega^{\frac{1}{2} \text{tr}(\alpha\gamma x^2)} |\alpha x\rangle \langle x| & \beta = 0 \\ \frac{\tilde{l}(-\beta)}{\sqrt{d}} \sum_{x,y \in \mathbb{F}_d} \omega^{\frac{1}{2} \text{tr}(\beta^{-1}(\alpha y^2 - 2xy + \delta x^2))} |x\rangle \langle y| & \beta \neq 0 . \end{cases} \quad (42)$$

Since

$$\frac{\tilde{l}(-\beta)}{\sqrt{d}} \notin \mathbb{Q} \quad \left(\frac{\tilde{l}(-\beta)}{\sqrt{d}} \right)^2 \in \mathbb{Q} \quad (43)$$

and since g_{θ_p} generates the Galois group, we must have

$$g_{\theta_p} \left(\frac{\tilde{l}(-\beta)}{\sqrt{d}} \right) = - \frac{\tilde{l}(-\beta)}{\sqrt{d}} , \quad (44)$$

implying

$$g_{\Delta_1} \left(\frac{\tilde{l}(-\beta)}{\sqrt{d}} \right) = (-1)^u \frac{\tilde{l}(-\beta)}{\sqrt{d}} . \quad (45)$$

In view of Lemma 1 in ref. [9], and our assumption that n is odd, we may write this in the form

$$g_{\Delta_1} \left(\frac{\tilde{l}(-\beta)}{\sqrt{d}} \right) = \frac{\tilde{l}(-\Delta_1 \beta)}{\sqrt{d}} . \quad (46)$$

Therefore

$$g_{\Delta_1}(U_{\bar{G}_2}) = \begin{cases} l(\alpha) \sum_{x \in \mathbb{F}_d} \omega^{\frac{1}{2} \text{tr}(\Delta_1 \alpha \gamma x^2)} |\alpha x\rangle \langle x| & \beta = 0 \\ \frac{\tilde{l}(-\Delta_1 \beta)}{\sqrt{d}} \sum_{x, y \in \mathbb{F}_d} \omega^{\frac{1}{2} \text{tr}(\Delta_1 \beta^{-1} (\alpha y^2 - 2xy + \delta x^2))} |x\rangle \langle y| & \beta \neq 0 . \end{cases} \quad (47)$$

Thus

$$g_{\Delta_1}(U_{\bar{G}_2}) = U_{K_{\Delta_1} \bar{G}_2 K_{\Delta_1}^{-1}} . \quad (48)$$

In view of the fact that the representation of $\text{SL}(2, \mathbb{F}_d)$ is faithful we can now deduce

$$U_{G_1} U_{G_2} = U_{G_1 G_2} . \quad (49)$$

implying that the representation of $\text{GL}_p(2, \mathbb{F}_d)$ is also faithful, as claimed. \square

As an immediate consequence of this one has

$$U_G^\dagger = U_{G^{-1}} \quad (50)$$

for all $G \in \text{GL}_p(2, \mathbb{F}_d)$. We also remark that if n is even then it follows from the above that

$$U_{G_1} U_{G_2} = \pm U_{G_1 G_2} \quad (51)$$

so that the representation is, in a sense, “close to faithful”.

It follows from Lemma 1 of ref. [9] that \mathbb{F}_p contains numbers which are quadratic non-residues with respect to the embedding field \mathbb{F}_d if and only if n is odd. Consequently, extending the Clifford group to include the full set of g -unitaries will give us the “missing” Möbius transformations discussed in Section 4 if and only if n is odd.

7. COMPLEMENTARITY POLYTOPES

To give a geometrical interpretation to the g -unitaries we place ourselves in the set of Hermitean matrices of unit trace, regarded as an Euclidean space equipped with the standard Hilbert-Schmidt metric. The set of density matrices forms a convex body within this space, which we call Bloch space. Usually one thinks of Bloch space as a $(d^2 - 1)$ -dimensional vector space, with its origin at the maximally mixed state. Anyway the distance between two Hermitean matrices M_1 and M_2 (be they density matrices or not) is

$$D(M_1, M_2) = \sqrt{\frac{1}{2} \text{Tr}(M_1 - M_2)^2} . \quad (52)$$

In this Bloch space we introduce a regular simplex with d^2 vertices represented by matrices A_{ij} , with each index ranging over d possible values. They are chosen to obey

$$\text{Tr} A_{ij} A_{kl} = \begin{cases} d & \text{if } (i, j) = (k, l) \\ 0 & \text{otherwise} . \end{cases} \quad (53)$$

At the end things will be arranged so that these matrices can be identified with Wootters' phase point operators [4], but at the outset they just define the vertices of a regular simplex in Bloch space. The simplex is centred at the maximally mixed state by insisting that

$$\sum_{i=0}^{d-1} \sum_{j=0}^{d-1} A_{ij} = d \mathbf{1} . \quad (54)$$

A facet of the simplex consists of all matrices M such that $\text{Tr} A_{ij} M = 0$ for some fixed choice of i, j .

Now suppose that d is a prime or a prime power, in which case there exists a combinatorial structure known as a finite affine plane with d^2 points. One defines subsets of points, known as lines, such that any pair of points belong to a unique line and such that for every point not belonging to a line there is a unique line disjoint from the given line and containing the given point. (This is the parallel postulate in affine geometry.) The most important theorems concerning finite affine planes state that [34]

- (1) the number of lines in the finite affine plane equals $d(d+1)$,
- (2) each line contains d points,
- (3) each point is contained in $d+1$ lines,
- (4) and there are altogether $d+1$ sets of d disjoint lines.

Therefore we can associate a unit trace operator (and hence a point in Euclidean space) to each given line by summing the d phase point operators contained in the line,

$$P_r^{(z)} = \frac{1}{d} \sum_{\mathbf{p} \in l_r^z} A_{\mathbf{p}} . \quad (55)$$

The index $z \in \{\mathbb{F}_d \cup \infty\}$ labels the $d+1$ pencils of parallel lines with the symbol ∞ used to label the “vertical” pencil, and the index $r \in \mathbb{F}_d$ labels the individual lines within such a pencil. There are $d(d+1)$ such line operators altogether.

Here we have tacitly identified the affine plane with the vector space $(\mathbb{F}_d)^2$. Actually affine planes not coordinatized by finite fields do exist and the use of a finite field for labelling purposes is not mandatory, but we will stick to it.

Due to eq. (54) and the fact that $d+1$ lines intersect in each point eq. (55) can be inverted to give

$$A_{\mathbf{p}} = \sum_{l_r^z \ni \mathbf{p}} P_r^{(z)} - \mathbf{1} . \quad (56)$$

The sum is over all the $d+1$ operators representing lines going through the point represented by A .

Using the combinatorics of the affine plane one easily checks that



FIGURE 1. A finite plane is formed by the operators A_{ij} . To the left we illustrate the equation $P_0^{(\infty)} = \frac{1}{3}(A_{00} + A_{01} + A_{02})$. To the right we illustrate the equation $A_{00} = P_0^{(0)} + P_0^{(1)} + P_0^{(2)} + P_0^{(\infty)} - \mathbf{1}$. Note that both the points and the lines in the affine plane correspond to points in Bloch space.

$$\text{Tr} P_r^{(z)} P_{r'}^{(z')} = \begin{cases} 1 & \text{if } z = z' \text{ and } r = r' & \text{(identical)} \\ 0 & \text{if } z = z' \text{ and } r \neq r' & \text{(no common point)} \\ \frac{1}{d} & \text{if } z \neq z' & \text{(one common point)} \end{cases} \quad (57)$$

The geometrical meaning of this, in Euclidean space, is that any collection of d parallel (non-intersecting) lines forms a regular simplex spanning a $(d-1)$ -dimensional plane, and that the collection of $d+1$ planes defined in this way are totally orthogonal. The convex hull of the $d(d+1)$ vertices defines the complementarity polytope [5].

A complementarity polytope exists in all Euclidean spaces of dimension $d^2 - 1$. What is special about d being a prime power is only that we were able to use the combinatorics of a finite affine plane to inscribe it in a regular simplex. To understand its full face structure we define generalized phase point operators by picking one matrix $P_r^{(z)}$ for each value of z , and summing them to obtain

$$A_{\vec{r}} = \sum_z P_r^{(z)} - \frac{1}{d} \mathbf{1} . \quad (58)$$

Unlike the summation in eq. (56), we are now allowing arbitrary choices of lines (label r) from the pencils (label z). The operators $A_{\vec{r}}$ have unit trace. There are d^{d+1} such operators altogether labelled by a vector \vec{r} with entries in \mathbb{F}_d . It is easy to see that

$$\text{Tr} A_{\vec{r}} P_r^{(z)} = \begin{cases} 1 & \text{if the } z\text{th component of } \vec{r} \text{ equals } r \\ 0 & \text{otherwise} \end{cases} . \quad (59)$$

Therefore the entire complementarity polytope is confined between pairs of parallel hyperplanes containing pairs of orthocomplemented faces. This includes faces that contain $d^2 - 1$ vertices spanning a facet associated to the phase point operator $A_{\vec{r}}$, and all facets arise in this way. The particular phase point operator simplex we started out with is just one out of d^{d-1} such simplices in which the complementarity polytope is inscribed, but we will soon see why it is useful to single out one of them for attention.

If $d = 2$ Bloch space has $2^2 - 1 = 3$ dimensions and the complementarity polytope is a regular octahedron with $2 \cdot 3$ vertices and 2^3 facets, inscribed in a regular

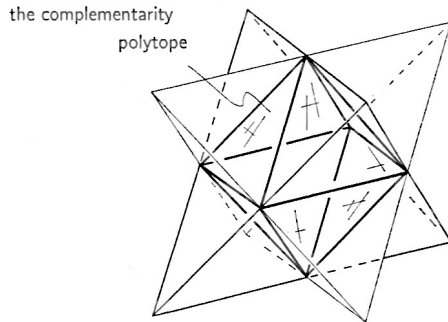


FIGURE 2. When $d = 2$ the complementarity polytope is inscribed in $2^{2-1} = 2$ simplices. It is a special property of $d = 2$ that these are on the same footing relative to the set of quantum states (the round Bloch sphere). Reprinted with permission [35].

tetrahedron. However, we will see that the even and odd dimensional cases differ a bit from the point of view of their symmetry groups.

8. THE SYMMETRY GROUP OF THE COMPLEMENTARITY POLYTOPE

The symmetry group of the complementarity polytope is the huge group

$$S_{d+1} \times S_d \times S_d \times \cdots \times S_d \in O(d^2 - 1) , \quad (60)$$

where S_d is the group of all permutations of the vertices of a $(d^2 - 1)$ -dimensional simplex, and S_{d+1} is the group of permutations of the $(d - 1)$ -planes. However, we will naturally insist that its vertices correspond to pure quantum states, which means that the polytope must be inscribed into the convex body of quantum states. Given that we have a complete set of MUB available this is achieved by

$$P_r^{(z)} = \Pi_r^{(z)} = |e_r^{(z)}\rangle \langle e_r^{(z)}| . \quad (61)$$

This is the tricky step, but we know it can be done if d is a power of a prime number. In any case the symmetry group of the body of quantum states is, ignoring reflections,

$$U(d)/U(1) \in SO(d^2 - 1) . \quad (62)$$

The intersection of the two groups is precisely the Clifford group modulo phase factors. Provided that d is an odd prime power this group is in fact

$$SL(2, \mathbb{F}_d) \ltimes (\mathbb{F}_d)^2 , \quad (63)$$

where \mathbb{F}_d is considered as a group under addition. Including anti-unitary symmetries leads to the extended Clifford group, but if $p > 3$ we have to include g -unitaries as well in order to have the full projective group of Möbius transformations acting on the label z . We would like to understand in geometrical terms what property of the complementarity polytope singles out this group for attention.

In the preceding section we showed that a complementarity polytope can be inscribed into a regular simplex (whose facets are suitably selected facets of the complementarity polytope) provided an affine plane of order d exists. The full

symmetry group of the simplex including reflections is the symmetric group S_{d^2} . We are interested in the largest common subgroup of the symmetry groups of the two polytopes. Provided d is a prime power, and provided the finite affine plane is coordinatized by a finite field \mathbb{F}_d of order d , the common subgroup is easily recognized. In the affine plane it must take points to points and lines to lines. This means that it is isomorphic to the affine group over the field in question, namely

$$\mathrm{GL}(2, \mathbb{F}_d) \ltimes (\mathbb{F}_d)^2, \quad (64)$$

This is almost, but not quite, the same answer as that obtained when one restricts oneself to those symmetries of the complementarity polytope that preserve also the inscribed body of density matrices, but if $d > 3$ it is not quite the same. When d is prime the g-unitaries provide all of the extra transformations. When d is a prime power they provide some of the extra transformations, namely those coming from the subgroup GL_p of GL .

Thus we have arrived at our geometric interpretation of the g-unitaries: When their action is restricted to the MUB vectors they are in fact rigid rotations in Bloch space. Harking back to the end of section 5 we note that this interpretation hinges on the fact that all the trace inner products between these vectors are rational numbers. Hence the interpretation of g-unitaries in terms of rotations in Bloch space has a very limited scope, and does not apply to their action on arbitrary vectors in their domain.

We note that if d is even there is a subtle difference. Consider $d = 2$. Then the two groups GL and SL coincide, and the group (64) is isomorphic to S_4 , the full symmetry group of the tetrahedron including reflections. Quantum mechanically this is realized by a combination of unitary and anti-unitary operations. The Clifford group modulo phases equals the symmetry group of the octahedron, which is again isomorphic to S_4 but subtly different from the group (63) [20]. Quantum mechanically it is realized by unitary operations, not all of which preserve the phase point operator simplex.

Concerning odd prime d we observe that the projective group PGL is a subgroup of the S_{d+1} factor of the group (60), which permutes the bases in the set of MUB. Moreover PGL has two components depending on whether the determinant Δ of the $\mathrm{GL}(2)$ matrix is a quadratic residue or not. If $p = 3$ modulo 4 it happens that -1 is a quadratic non-residue, which means that the full set of projective transformations is recovered from the extended symplectic group (which is realized by unitary and anti-unitary transformations). If on the other hand $p = 1$ modulo 4 then -1 is a quadratic residue, and one needs to consider general g-unitaries to recover the full set of projective transformations. For prime power dimensions $d = p^n$ the situation is more complicated, and we do not obtain all of the projective transformations from the g-unitaries unless n is odd.

We also note that, given the identification in eq. (61), there exists a special set of phase point operators taking a simple form. When d is odd this includes the parity operator

$$A = \frac{1}{d} \sum_{\mathbf{u}} D_{\mathbf{u}}, \quad (65)$$

where the sum extends over the finite phase space. The full set of special phase point operators consists of

$$A_{\mathbf{p}} = D_{\mathbf{p}} A D_{\mathbf{p}}^{-1} = \sum_{\mathbf{u}} \omega^{2\Omega(\mathbf{p}, \mathbf{u})} D_{\mathbf{u}}. \quad (66)$$

Thus the special phase point operator simplex is a unitary operator basis obtained in a simple way from the unitary operator basis provided by the Weyl-Heisenberg group. The generalized phase point operators defined by eq. (58) can also be collected into simplices, transforming into each other under the symplectic group [36], but their spectral properties are not as attractive [28, 37].

9. MUB-CYCLERS

We now come to the question of whether the projective permutations of the bases include MUB-cyclers, that is transformations that cycle through all bases in succession. Thus we need an element of PGL of order $d + 1$, or equivalently an element of GL of suborder $d + 1$ (the definition of suborder will be elaborated in the next paragraph). In odd prime dimensions the existence of such an element G is sufficient, but in the case of odd prime power dimensions $d = p^n$, where entries of G belong to the field \mathbb{F}_{p^n} we need an extra condition, that the determinant of G belongs to the prime field \mathbb{F}_p (i.e. it is an integer modulo p) so that G admits a g -unitary representation U_G as in section 5. In the following theorems, we establish that MUB-cycling g -unitaries exist if and only if the exponent n is odd, and provide a characterization of every possible MUB-cycler for when n is odd.

It follows from the expression for the Möbius transformation (17) that if

$$G = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}, \quad G^m = \begin{pmatrix} \alpha_m & \beta_m \\ \gamma_m & \delta_m \end{pmatrix} \quad (67)$$

then U_{G^m} takes $z = 0$ to $z = \beta_m/\delta_m$ (respectively ∞) if $\delta_m \neq 0$ (respectively $\delta_m = 0$). So if we define the suborder of G to be the smallest positive integer m for which $\beta_m = 0$, then U_{G^m} will take a MUB basis back to itself. However, as U_{G^m} can permute the vectors in the basis, m need not be the order of G . Generally, the suborder is a factor of the order of G . Following Lemma 3, we will see that the smallest positive integer m such that G^m is proportional to the 2×2 identity matrix is an equivalent definition of the suborder of G .

Let G be an element of $\text{GL}(2, \mathbb{F}_d)$ with trace t and determinant $\Delta \neq 0$. The eigenvalues of G are roots of the characteristic polynomial $x^2 - tx + \Delta = 0$ and are given by

$$\lambda_{\pm} = (t \pm \sqrt{t^2 - 4\Delta})/2. \quad (68)$$

If $t^2 - 4\Delta$ is zero or a quadratic residue, i.e. it has a non-zero square root in \mathbb{F}_d , then λ_{\pm} belong to the field \mathbb{F}_d . Otherwise, the eigenvalues do not belong to \mathbb{F}_d , but they are still well-defined, and we can extend the field \mathbb{F}_d to \mathbb{F}_{d^2} to include them. To deal with these cases, it is convenient for us to classify GL elements into three types, as summarized in the table below.

Throughout this section, we will assume that the dimension d is a prime power of the form $d = p^n$, where p is an odd prime number.

Theorem 2. *Let G be an element of $\text{GL}(2, \mathbb{F}_d)$ with determinant Δ .*

1. *If G is of type 1, then G has suborder of at most $d - 1$.*
2. *If G is of type 2, then G has suborder of at most $d + 1$ and satisfies*

$$G^{d+1} = \Delta I \quad (I \text{ is the } 2 \times 2 \text{ identity matrix}). \quad (69)$$

	Definition in terms of t and Δ	Equivalent definition
Type 1	$t^2 - 4\Delta$ is a quadratic residue	$\lambda_{\pm} \in \mathbb{F}_d, \lambda_+ \neq \lambda_-$
Type 2	$t^2 - 4\Delta$ is a quadratic non-residue	$\lambda_{\pm} \notin \mathbb{F}_d, \lambda_+ \neq \lambda_-$
Type 3	$t^2 - 4\Delta = 0$	$\lambda_{\pm} = t/2 \in \mathbb{F}_d$

TABLE 1. A classification of GL elements into three types, among which only type 2 can give rise to MUB-cyclers, as will be seen in the following theorems.

3. If G is of type 3, then G has suborder of at most d .

We will start with the following lemma, whose proof by induction can be carried out straightforwardly. We will leave the proof for our readers.

Lemma 3. *Let A be any 2×2 matrix, with trace t and determinant Δ . For any integer $m \geq 1$, it holds that*

$$A^m = s_m A - s_{m-1} \Delta I, \quad (70)$$

where the sequence $\{s_m\}$ is defined by the recurrence relation

$$s_{m+1} = t s_m - \Delta s_{m-1}, \quad (71)$$

with $s_0 = 0$ and $s_1 = 1$. Equivalently, s_m can be calculated by

$$s_m = \begin{cases} (\lambda_+^m - \lambda_-^m)/(\lambda_+ - \lambda_-) & \text{if } \lambda_+ \neq \lambda_- \\ m\lambda_+^{m-1} & \text{if } \lambda_+ = \lambda_- \end{cases} \quad (72)$$

where λ_{\pm} are roots of the characteristic polynomial $x^2 - tx + \Delta$.

Remark. If $A = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$, we can explicitly rewrite Eq. (70) as

$$A^m = \begin{pmatrix} s_m \alpha - s_{m-1} \Delta & s_m \beta \\ s_m \gamma & s_m \delta - s_{m-1} \Delta \end{pmatrix} \quad (73)$$

and see that if $\beta \neq 0$ (for a non-zero determinant, we can always make $\beta \neq 0$ using the form in Eq. (89) at the end of this section) the suborder of A is the smallest positive integer m for which $s_m = 0$. A^m is then proportional to the identity matrix.

Proof of Theorem 2.

Let λ_{\pm} be the eigenvalues of G , based upon which we define a sequence $\{s_m\}$ just as in (72). Note that although λ_{\pm} might not be in the field \mathbb{F}_d , the sequence $\{s_m\}$ always is, as can be seen from the recursive definition in (71). Lemma 3 implies that if $s_m = 0$ for some m , then $G^m = -s_{m-1} \Delta I$, and therefore the suborder of G is at most m . Let us now consider specific cases. Facts about finite fields in Section 3 will be used implicitly here.

1. If G is of type 1, then the eigenvalues λ_{\pm} are in \mathbb{F}_d , and therefore

$$\lambda_+^{d-1} = \lambda_-^{d-1} = 1, \quad (74)$$

which implies

$$s_{d-1} = (\lambda_+^{d-1} - \lambda_-^{d-1})/(\lambda_+ - \lambda_-) = 0. \quad (75)$$

Therefore G has suborder of at most $d - 1$.

2. If G is of type 2, we create an extension field \mathbb{F}_{d^2} from the base field \mathbb{F}_d and the generator $j \equiv \sqrt{t^2 - 4\Delta}$. Since $(j^d)^2 = (t^2 - 4\Delta)^d = t^2 - 4\Delta = j^2$, we have $j^d = \pm j$. Because j is not in the field \mathbb{F}_d we cannot have $j^d = j$, and it therefore must be the case that $j^d = -j$. As d is odd, we have

$$\lambda_{\pm}^d = \frac{(t \pm j)^d}{2^d} = \frac{t \pm j^d}{2} = \frac{t \mp j}{2} = \lambda_{\mp}. \quad (76)$$

We then use (72) to derive

$$s_d = (\lambda_+^d - \lambda_-^d)/(\lambda_+ - \lambda_-) = -1, \quad (77)$$

$$s_{d+1} = \frac{\lambda_+^{d+1} - \lambda_-^{d+1}}{\lambda_+ - \lambda_-} = \frac{\lambda_+ \lambda_- - \lambda_- \lambda_+}{\lambda_+ - \lambda_-} = 0, \quad (78)$$

and therefore

$$G^{d+1} = s_{d+1}G - s_d\Delta I = \Delta I. \quad (79)$$

It follows that G has suborder of at most $d+1$.

3. If G is of type 3, then $\lambda_{\pm} = t/2$. It follows from Eq. (72) that $s_d = d\lambda_+^{d-1} = 0$, so G has suborder of at most d .

□

Lemma 4. Let $G \in \text{GL}_p(2, \mathbb{F}_d)$, i.e. an element $\text{GL}(2, \mathbb{F}_d)$ whose determinant Δ is in \mathbb{F}_p . Let $\bar{\theta}$ be a primitive element of \mathbb{F}_{d^2} . Note that $(d-1)/(p-1)$ is an integer, so we can define an element $\eta \in \mathbb{F}_{d^2}$ as

$$\eta \equiv \bar{\theta}^{(d-1)/(p-1)}. \quad (80)$$

Then G is of type 2 if and only if it has eigenvalues η^r and η^{dr} , for some integer r not a multiple of $(d+1)/2$ in the range $0 < r < (p-1)(d+1)$.

Proof. Assume that G is of type 2, and let λ_{\pm} be its eigenvalues. Following Eq. (76) in the proof of Theorem 2 we have $\lambda_{\pm}^d = \lambda_{\mp}$, so we may write

$$\lambda_+ = \bar{\theta}^k \quad \lambda_- = \bar{\theta}^{dk} \quad (81)$$

for some integer k in the range $1 \leq k \leq d^2 - 1$. The fact that the eigenvalues are not in \mathbb{F}_d means that k is not a multiple of $d+1$. The fact that $\Delta \in \mathbb{F}_p$ means

$$\bar{\theta}^{pk(d+1)} = \Delta^p = \Delta = \bar{\theta}^{k(d+1)} \quad (82)$$

So

$$\bar{\theta}^{k(p-1)(d+1)} = 1 \quad (83)$$

implying that $(d-1) \mid k(p-1)$. Let $r = k(p-1)/(d-1)$. Then the eigenvalues can be written as

$$\lambda_+ = \eta^r \quad \lambda_- = \eta^{dr}. \quad (84)$$

The requirement that $\lambda_{\pm} \notin \mathbb{F}_d$ means

$$\eta^{dr} = \bar{\theta}^{rd(d-1)/(p-1)} \notin \mathbb{F}_d \quad (85)$$

which is true if and only if r is not a multiple of $(d+1)/2$.

Conversely, if G has eigenvalues of the form $\lambda_+ = \eta^r$ and $\lambda_- = \eta^{dr}$, where r is not a multiple of $(d+1)/2$, then λ_{\pm} are not in the field \mathbb{F}_d , and G is therefore of type 2. One can further verify that its trace is in \mathbb{F}_d and its determinant is in \mathbb{F}_p by defining

$$t \equiv \eta^r + \eta^{dr} \quad \Delta \equiv \eta^{(d+1)r} \quad (86)$$

and using the facts $\eta^{d^2} = \eta$ and $\eta^{(d+1)p} = \eta^{d+1}$ to check that

$$t^d = t \quad \Delta^p = \Delta. \quad (87)$$

□

With Lemma 4, all type-2 elements of $\text{GL}(2, \mathbb{F}_d)$ whose determinant Δ is in \mathbb{F}_p (this extra condition is to guarantee the feasibility of their g-unitary representation) can now be characterized by an integer r , via their eigenvalues. In the next theorem, we will pin down which exact values of r correspond to MUB-cyclers.

Theorem 5. *Let $G \in \text{GL}_p(2, \mathbb{F}_d)$ be of type 2 and let the integer r be as in the statement of Lemma 4.*

1. *When n is even, G has suborder of at most $(d+1)/2$.*
2. *When n is odd, G has suborder $d+1$ if and only if $\gcd(r, d+1) = 1$.*

Proof. Let λ_{\pm} be the eigenvalues of G and the sequence s_m be as defined in Lemma 3. We recall that the suborder of G is the smallest positive integer m for which $s_m = 0$, which is equivalent to $\lambda_+^m = \lambda_-^m$ in this case when the two eigenvalues are distinct because G is of type 2.

- (1) When n is even, $(d-1)/(p-1) = 1 + p + \dots + p^{n-1}$ is an even integer, so $(d-1)/2$ is a multiple of $(p-1)$. It then follows from Eq. (83) that

$$\bar{\theta}^{k(d-1)(d+1)/2} = 1, \quad (88)$$

which implies $\lambda_+^{(d+1)/2} = \lambda_-^{(d+1)/2}$, or $s_{(d+1)/2} = 0$. Therefore G has suborder of at most $(d+1)/2$ and cannot be a MUB-cycler.

- (2) When n is odd, $(d-1)/(p-1)$ is an odd integer. It follows from this, and the fact that $\gcd(d+1, d-1) = 2$, that $(d-1)/(p-1)$ is co-prime to $d+1$. We have $\lambda_+^m = \lambda_-^m$ if and only if $\eta^{m(d-1)r} = 1$, which in turn is true if and only if $mr(d-1)/(p-1)$ is a multiple of $d+1$. Therefore G has suborder $d+1$ if and only if r is co-prime to $d+1$.

□

In summary, in this section we have proved the nonexistence of MUB-cyclers when the exponent n is even. In the case n is odd, we have identified all elements in $\text{GL}(2, \mathbb{F}_d)$ that give rise to MUB-cyclers according to the characteristics of their eigenvalues. Lastly, we want to provide an explicit form for these MUB-cycling elements. The proof in the Appendix of ref. [28] can be easily extended to show that for any element in $G \in \text{GL}(2, \mathbb{F}_d)$ with trace t and determinant Δ , where $t^2 - 4\Delta \neq 0$, there exists $S \in \text{SL}(2, \mathbb{F}_d)$ such that

$$G = SG_c S^{-1} \quad G_c = \begin{pmatrix} 0 & -\Delta \\ 1 & t \end{pmatrix}. \quad (89)$$

Therefore, an element of $\text{GL}(2, \mathbb{F}_d)$ is a MUB-cycling matrix if and only if it is conjugate to G_0^r where

$$G_0 = \begin{pmatrix} 0 & -\eta^{(d+1)} \\ 1 & \eta + \eta^d \end{pmatrix}, \quad (90)$$

η is defined as in Eq. (80), and r is an integer co-prime to $d+1$. Note that the order of G_0 is $(p-1)(d+1)$ (because this is the smallest integer r such that η^r and η^{dr} , the eigenvalues of G_0^r are both equal to 1).

It follows from this that anti-symplectic MUB-cycling matrices exist if and only if $d = 3 \pmod{4}$ (a fact already shown in ref. [9]). In fact G_0^r is anti-symplectic if and only if $\eta^{r(d+1)} = -1$, which in turn is true if and only if r is an odd multiple of $(p-1)/2$. If $d = 1 \pmod{4}$ then $(p-1)/2$ is even and so no multiple of $(p-1)/2$ is co-prime to $d+1$. If, on the other hand, $d = 3 \pmod{4}$ it is easily seen that $(p-1)/2$ is co-prime to $d+1$ implying that $G_0^{r(p-1)/2}$ is a MUB-cycling anti-symplectic for every r co-prime to $d+1$.

10. EIGENVECTORS OF MUB-CYCLERS

We now come to the question of finding the eigenvectors of a g -unitary. The result we prove will play a crucial role in our construction in the next section, of MUB-balanced states in prime power dimensions equal $3 \pmod{4}$.

An ordinary unitary is, of course, always diagonalizable. However the situation with g -unitaries is more vexed. Indeed, it is not guaranteed that an arbitrary g -unitary will have any eigenvectors at all. This is easily seen in the special case of an anti-unitary. Suppose that U is an anti-unitary and $|\psi\rangle$ an eigenvector, so that

$$U|\psi\rangle = \lambda|\psi\rangle \quad (91)$$

for some λ . It follows from the definition of the adjoint of an anti-unitary [1] that

$$|\lambda|^2 \langle \psi, \psi \rangle = \langle U\psi, U\psi \rangle = (\langle \psi, U^\dagger U\psi \rangle)^* = \langle \psi, \psi \rangle \quad (92)$$

(where we have temporarily switched from Dirac notation to the notation usual in pure mathematics). So λ is a phase. Consequently

$$U^2|\psi\rangle = |\lambda|^2|\psi\rangle = |\psi\rangle. \quad (93)$$

We conclude that $|\psi\rangle$ must be an eigenvector of the unitary U^2 with eigenvalue 1. It follows that if U^2 does not have any eigenvectors with eigenvalue 1, then U does not have any eigenvectors at all. For an example consider the anti-unitary U which acts on \mathbb{C}^2 according to

$$U \begin{pmatrix} x \\ y \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} y^* + x^* \\ y^* - x^* \end{pmatrix}. \quad (94)$$

Since the eigenvalues of U^2 are $\pm i$, U has no eigenvectors.

As we will see analogous statements hold in the case of an arbitrary g -unitary (except that in the case of a g -unitary which is not an anti-unitary the eigenvalues, if they exist, do not have to be phases).

Another important difference between g -unitaries and ordinary unitaries is that multiplying an eigenvector by a scalar can change the eigenvalue. Again, this is most easily seen in the special case of an anti-unitary. Thus, if U is an anti-unitary and $|\psi\rangle$ an eigenvector with eigenvalue $e^{i\theta}$, then $e^{i\phi}|\psi\rangle$ is an eigenvector with eigenvalue $e^{i(\theta-2\phi)}$. This means, in particular, that in the case of an anti-unitary, if one adjusts the overall phase appropriately, one can always ensure that the eigenvalue is 1.

Wigner [1] analyzed the eigenvectors of anti-unitaries. The problem of extending his analysis to the case of an arbitrary g -unitary is not straightforward. In this section we will confine ourselves to g -unitaries of a very special kind: namely the MUB-cycling g -unitaries defined in the last section. For such g -unitaries it is not difficult to give a complete characterization of their eigenvectors and eigenvalues. The results of our analysis are summarized in theorem 6. The theorem states that g -unitaries of the kind we consider always do have eigenvectors. Moreover, their

eigenvectors are confined to a one dimensional subspace (in other words, they are unique, up to multiplication by a scalar). Finally, as with anti-unitaries, one can always adjust the overall scale factor so as to ensure that the eigenvalue is 1.

Theorem 6. *Let $d = p^n$ where n is odd, and let $G \in \text{GL}_p(2, \mathbb{F}_d)$ be a MUB-cycler. Let V_c be the subspace of the full Hilbert space consisting of all vectors whose standard basis components are in the cyclotomic field $\mathbb{Q}(\omega)$. Then*

(1) *There exists a non-zero vector $|\psi\rangle \in V_c$ such that*

$$U_G|\psi\rangle = |\psi\rangle . \quad (95)$$

(2) *Let $|\phi\rangle \in V_c$ be arbitrary. Then $|\phi\rangle$ is an eigenvector of U_G if and only if $|\phi\rangle = \mu|\psi\rangle$ for some $\mu \in \mathbb{Q}(\omega)$.*

(3) *Let $2m_0$ be the smallest positive integer (necessarily even) such that $U_G^{2m_0}$ is unitary. Then the eigenspace of $U_G^{2m_0}$ with eigenvalue 1 is one-dimensional and is spanned by $|\psi\rangle$.*

(4) *$|\psi\rangle$ is an eigenvector of the parity operator, with eigenvalue $(-1)^{\frac{p-1}{2}}$.*

Remark. The vector $|\psi\rangle$ cannot be assumed to be normalized (since it may happen that the normalization constant is not in the field $\mathbb{Q}(\omega)$, and since, even if it is in the field, the normalized vector may not have eigenvalue 1).

The fact that the eigenvectors of U_G are also eigenvectors of $U_G^{2m_0}$ with eigenvalue 1 is important as it provides us with a means of calculating them.

As a special case of this theorem, the MUB cycling anti-unitaries, whose existence in dimension $d = 3 \pmod 4$ was established in ref. [9], all have eigenvectors which are unique up to scalar multiplication.

In the fourth statement of the theorem, the term “parity operator” refers to the unitary

$$A = \sum_x | -x\rangle\langle x| \quad (96)$$

which already appeared in eq. (65). It follows from eq. (12) and Lemma 1 in ref. [9] that

$$A = (-1)^{\frac{p-1}{2}} U_P \quad \text{where} \quad P = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} . \quad (97)$$

Proof. Immediate consequence of Lemmas 7–10 proved below. \square

In the remainder of this section it will always be assumed that the exponent n is odd, and that G is a fixed element of $\text{GL}_p(2, \mathbb{F}_d)$ with eigenvalues η^r and η^{rd} (as in Lemma 4 and Theorem 5) where r is co-prime to $d+1$ so that G is a MUB-cycler. We will always write $t = \text{Tr}(G)$ and $\Delta = \det(G)$. From Theorem 2 we know

$$G^{d+1} = \Delta I . \quad (98)$$

So, if we define the multiplicative order of Δ to be the smallest positive integer m such that $\Delta^m = 1$, this is the same as the smallest positive integer m such that U_G^m is a unitary. We have that $\Delta^m = 1$ if and only if $mr(d^2 - 1)/(p - 1)$ is a multiple of $d^2 - 1$, which, in turn, is true if and only if mr is a multiple of $p - 1$. Since r is odd this means that the multiplicative order of Δ must be even. We will therefore denote it $2m_0$.

Lemma 7. *With notations and definitions as above, suppose $|\phi\rangle \in V_c$ is an eigenvector of U_G so that*

$$U_G|\phi\rangle = \lambda|\phi\rangle \quad (99)$$

for some $\lambda \in \mathbb{Q}(\omega)$. Then

$$U_G^{2m_0}|\phi\rangle = |\phi\rangle . \quad (100)$$

Proof. The fact that $\Delta^{2m_0} = 1$ implies $\Delta^{m_0} = \pm 1$. Since $2m_0$ is the smallest positive integer m such that $\Delta^m = 1$ we must in fact have $\Delta^{m_0} = -1$. This means that $U_G^{m_0}$ is an anti-unitary. Repeatedly applying U_G to Eq. (99) gives

$$U_G^{m_0}|\psi\rangle = \kappa|\psi\rangle \quad (101)$$

where $\kappa = \lambda g_\Delta(\lambda) \dots g_\Delta^{m_0-1}(\lambda)$. It then follows from the discussion at the beginning of this section [see Eq. (93)] that

$$U_G^{2m_0}|\psi\rangle = |\psi\rangle . \quad (102)$$

□

Lemma 8. *With notations and definitions as above, let \mathcal{S}_1 be the eigenspace of $U_G^{2m_0}$ corresponding to the eigenvalue 1. Then $\dim(\mathcal{S}_1) = 1$.*

Proof. Taking account of the fact that the representation of $\text{SL}(2, \mathbb{F}_d)$ is faithful, the projector onto \mathcal{S}_1 is

$$P_1 = \frac{1}{d+1} \sum_{u=0}^d U_{G^{2m_0 u}} , \quad (103)$$

implying

$$\dim \mathcal{S}_1 = \frac{1}{d+1} \sum_{u=0}^d \text{Tr}(U_{G^{2m_0 u}}) . \quad (104)$$

We will use a slightly improved version of Theorem 5 of ref. [9] to evaluate this expression. The original theorem states that given $S \in \text{SL}(2, \mathbb{F}_d)$ of the form

$$S = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \quad (105)$$

with determinant 1 and trace $t \neq 2$, then

$$\text{Tr}(U_S) = \begin{cases} l(t-2) & \beta \neq 0 \\ l(\alpha) & \beta = 0 . \end{cases} \quad (106)$$

We improve this statement by showing that for $t \neq 2$, $\text{Tr}(U_S) = l(t-2)$ irrespective of whether $\beta = 0$. Indeed, consider the case $\beta = 0$, when S takes the form

$$S = \begin{pmatrix} \alpha & 0 \\ \gamma & -\alpha^{-1} \end{pmatrix} . \quad (107)$$

We have $t = \alpha + \alpha^{-1}$ and $l(t-2) = l((\alpha^{1/2} - \alpha^{-1/2})^2)$. Note that

$$(\alpha^{1/2} - \alpha^{-1/2})^d = \alpha^{d/2} - \alpha^{-d/2} = \begin{cases} \alpha^{1/2} - \alpha^{-1/2} & \alpha^{1/2} \in \mathbb{F}_d \\ \alpha^{-1/2} - \alpha^{1/2} & \alpha^{1/2} \notin \mathbb{F}_d , \end{cases} \quad (108)$$

which means $\alpha^{1/2} - \alpha^{-1/2} \in \mathbb{F}_d$ if and only if $\alpha^{1/2} \in \mathbb{F}_d$, and consequently $l(\alpha) = l(t-2)$, as we wish to show.

Back to evaluating $\text{Tr}(G^{2m_0u})$, since G has eigenvalues η^r and η^{dr} , we have

$$\begin{aligned}\text{Tr}(G^{2m_0u}) &= \eta^{2rm_0u} + \eta^{2drm_0u} \\ &= \eta^{2rm_0u} + \eta^{-2rm_0u} \\ &= (\eta^{rm_0u} - \eta^{-rm_0u})^2 + 2\end{aligned}\tag{109}$$

where in the penultimate step we used the fact that $\eta^{2drm_0} = \eta^{-2rm_0}$ (as follows from the fact that $\Delta^{2m_0} = 1$). We see from this that $\text{Tr}(G^{2m_0u}) = 2$ if and only if $2rm_0u$ is a multiple of $(p-1)(d+1)$. Since $2m_0$ is the order of an element in a group of order $p-1$ it must be a divisor of $p-1$. So the condition is equivalent to the statement that ru is a multiple of $(d+1)(p-1)/2m_0$, which, in view of the fact that r is co-prime to $d+1$, is in turn equivalent to the statement that u is a multiple of $(d+1)$. Since u is in the range $0 \leq u \leq d$ this means $\text{Tr}(G^{m_0u}) = 2$ if and only if $u = 0$. Applying the improved result from Theorem 5 of ref. [9] we find

$$\text{Tr}(U_{G^{2m_0u}}) = \begin{cases} d & u = 0 \\ l((\eta^{rm_0u} - \eta^{-rm_0u})^2) & 1 \leq u \leq d \end{cases}\tag{110}$$

To evaluate the quadratic characters we again appeal to the fact that $\Delta^{m_0} = -1$, which implies $\eta^{drm_0} = -\eta^{-rm_0}$, so that

$$\begin{aligned}(\eta^{rm_0u} - \eta^{-rm_0u})^d &= \eta^{drm_0u} - \eta^{-drm_0u} \\ &= (-1)^{u+1}(\eta^{rm_0u} - \eta^{-rm_0u}).\end{aligned}\tag{111}$$

So for $1 \leq u \leq d$, we have

$$l((\eta^{rm_0u} - \eta^{-rm_0u})^2) = (-1)^{u+1}\tag{112}$$

and, consequently,

$$\dim \mathcal{S}_1 = \frac{1}{d+1} \left(d + \sum_{u=1}^d (-1)^{u+1} \right) = 1.\tag{113}$$

□

Lemma 9. *With definitions and notations as above there exists a non-zero vector $|\psi\rangle \in V_c$ such that*

$$U_G|\psi\rangle = |\psi\rangle.\tag{114}$$

Proof. We know from Lemma 8 that $\det(U_{G^{2m_0}} - I) = 0$. Since the matrix elements of $U_{G^{2m_0}}$ are in the field $\mathbb{Q}(\omega)$ we have, by a basic fact of linear algebra [39], that there exists a vector $|\phi\rangle \in V_c$ such that $U_{G^{2m_0}}|\phi\rangle = |\phi\rangle$. Since

$$U_{G^{2m_0}}U_G|\phi\rangle = U_G|\phi\rangle\tag{115}$$

and since \mathcal{S}_1 is one dimensional, we must have

$$U_G|\phi\rangle = \lambda|\phi\rangle\tag{116}$$

for some $\lambda \in \mathbb{Q}(\omega)$ such that

$$\lambda g_\Delta(\lambda) \dots g_\Delta^{2m_0-1}(\lambda) = 1.\tag{117}$$

It now follows from a variant of the proof of Hilbert's theorem 90 (see, for example, refs. [24] or [25] [we cannot make a direct application of the theorem since g_Δ may not generate the full Galois group]) that there exists $\mu \in \mathbb{Q}(\omega)$ such that

$$\lambda = \frac{\mu}{g_\Delta(\mu)} . \quad (118)$$

In fact, define a mapping $T: \mathbb{Q}(\omega) \rightarrow \mathbb{Q}(\omega)$ by

$$T(x) = \sum_{j=0}^{2m_0-1} a_j g_\Delta^j(x) \quad (119)$$

where

$$a_0 = 1, \quad a_1 = \lambda, \quad a_2 = \lambda g_\Delta(\lambda), \quad \dots, \quad a_{2m_0-1} = \lambda g_\Delta(\lambda) \dots g_\Delta^{2m_0-2}(\lambda) . \quad (120)$$

We have

$$\lambda g_\Delta(T(x)) = T(x) \quad (121)$$

for all x . Since the coefficients a_j are non-zero it follows from the Dedekind independence theorem (see, for example, refs. [24] or [25]) that there exists $x \in \mathbb{Q}(\omega)$ such that $T(x) \neq 0$. Choose such a value of x and set $\mu = T(x)$. Eq. (118) then follows. If we now define $|\psi\rangle = \mu|\phi\rangle$ we will have

$$U_G|\psi\rangle = |\psi\rangle . \quad (122)$$

□

Lemma 10. *With definitions and notations as above, let $|\phi\rangle$ be such that $U_G|\phi\rangle = \lambda|\phi\rangle$ for some λ . Then $|\phi\rangle$ is an eigenvector of the parity operator, with eigenvalue $(-1)^{\frac{p-1}{2}}$.*

Proof. It follows from Lemmas 7 and 8 that

$$U_G^{2m_0}|\phi\rangle = |\phi\rangle . \quad (123)$$

Since $\Delta^{2m_0} = 1$, and since $2m_0$ is the smallest positive integer for which that is the case, we must have $\Delta^{m_0} = -1$. In view of Eqs. (97) and (98) this means

$$\begin{aligned} A|\phi\rangle &= (-1)^{\frac{p-1}{2}} (U_G^{d+1})^{m_0} |\phi\rangle \\ &= (-1)^{\frac{p-1}{2}} (U_G^{2m_0})^{\frac{d+1}{2}} |\phi\rangle \\ &= (-1)^{\frac{p-1}{2}} |\phi\rangle . \end{aligned} \quad (124)$$

□

We conclude this section by observing that since every cycling g -unitary has exactly one eigenvector up to scalar multiplication, then $|\psi\rangle$ is an eigenvector of $U_{G_0^r}$ if and only if it is an eigenvector of U_{G_0} (where G_0 is the matrix defined by Eq. (90)). In view of the discussion at the end of Section 9 this means that the vectors which are eigenvectors of a cycling g -unitary form a single orbit under the extended Clifford group.

11. MUB-BALANCED STATES

We saw in the last section that if d is an odd power of p then every MUB-cycling g -unitary has an eigenvector. We now show that if, in addition, $d = 3 \pmod{4}$ then these eigenvectors are “rotationally invariant” or “MUB-balanced” states [7, 12, 10] and, *a fortiori*, minimum uncertainty states [7, 11, 12, 10]. The question as to what happens when $d = 1 \pmod{4}$ remains open.

Given a MUB, a MUB-balanced state is one for which the probabilities with respect to each basis are permutations of each other. In other words a normalized state $|\psi\rangle$ is MUB-balanced if and only if for all z there is a permutation f_z such that

$$p_j^{(z)} = p_{f_z(j)}^0 \quad (125)$$

where

$$p_j^{(z)} = |\langle e_j^{(z)} | \psi \rangle|^2 \quad (126)$$

This definition was introduced by Wootters and Sussman [7], who also showed that such states exist in every even prime power dimension. Wootters and Sussman went on to show that MUB-balanced states are minimum uncertainty states. Since it is central to this section it is worth summarizing their argument. Let

$$H_z = -\log_2 \left(\sum_j (p_j^{(z)})^2 \right) \quad (127)$$

be the quadratic Rényi entropy in basis z , and let $T = \sum_z H_z$ be the total entropy. Then can be shown that T satisfies the inequality

$$T \geq (d+1) \log_2 \left(\frac{d+1}{2} \right) \quad (128)$$

A minimum uncertainty state is one for which the bound is saturated. The necessary and sufficient condition for that to be true is that

$$\sum_j (p_j^{(z)})^2 = \frac{2}{d+1} \quad (129)$$

In a MUB-balanced state the fact that probabilities in each basis are the same up to permutation in every basis means that the sum $\sum_j (p_j^{(z)})^2$ is independent of z . In view of the identity

$$\sum_{j,z} (p_j^{(z)})^2 = 2 \quad (130)$$

it follows that Eq. (129) is satisfied, and that the state is consequently a minimum uncertainty state.

Our result is described by the following theorem

Theorem 11. *Suppose $d = 3 \pmod{4}$, and suppose $G \in \text{GL}_p(2, \mathbb{F})$ is a MUB-cycler. Let $2m_0$ be the integer defined in Theorem 1 of Section 9, and let $|\phi\rangle$ be a normalized eigenvector of the unitary $U_G^{2m_0}$ with eigenvalue 1. Then $|\phi\rangle$ is MUB-balanced.*

Remark. Note that, although the g -unitary U_G plays an essential role in the proof, a knowledge of the ordinary unitary $U_{G^{2m_0}}$ is sufficient if one only wants to calculate the state.

Since the eigenstates of MUB-cycling g -unitaries form a single orbit of the extended Clifford group we could restrict our attention to the eigenstates of MUB-cycling anti-unitaries. This would make the proof a little easier. Nevertheless, we

have chosen to give the proof for the case of an arbitrary g -unitary because this enables one to see why it fails when $d = 1 \pmod{4}$.

Proof. We know from Theorem 1 of Section 9 that there exists a state $|\psi\rangle$ such that

$$U_G|\psi\rangle = |\psi\rangle \quad (131)$$

Since U_G is a cycling g -unitary the set $\{z_0, z_1, \dots, z_d\}$ is a permutation of the full set of labels $\{0, 1, \dots, d-1, \infty\}$. We may write

$$|e_j^{z_l}\rangle = s_{l,j}\omega^{k_{l,j}}U_G^l|e_{p_l(j)}^0\rangle \quad (132)$$

where p_l is an l -dependent permutation, $k_{l,j}$ is an l and j -dependent integer and $s_{l,j}$ is an l and j -dependent sign. So

$$\begin{aligned} \langle e_{p_l(j)}^{z_l}|\psi\rangle &= s_{l,j}\omega^{-k_{l,j}}g_\Delta^{-l}\left(\langle e_j^0|U_j^\dagger|\psi\rangle\right) \\ &= s_{l,j}\omega^{-k_{l,j}}g_\Delta^{-l}\left(\langle e_j^0|\psi\rangle\right) \end{aligned} \quad (133)$$

Now let

$$F = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \quad (134)$$

be arbitrary. We have

$$K_\Delta F K_\Delta^{-1} = \begin{pmatrix} \alpha & \Delta^{-1}\beta \\ \Delta\gamma & \delta \end{pmatrix} \quad (135)$$

Since $\Delta = \theta^{\frac{r(d-1)}{(p-1)}}$, and since $r(d-1)/(p-1)$ is odd, Δ is a quadratic non-residue. The assumption that $d = 3 \pmod{4}$, together with Lemma 1 of ref. [9], means that -1 is also a quadratic non-residue. So there exists $x \in \mathbb{F}_d$ such that $x^2 = -\Delta$. If we define

$$S = \begin{pmatrix} x^{-1} & 0 \\ 0 & x \end{pmatrix} \quad (136)$$

then

$$SK_{-1}FK_{-1}S^{-1} = K_\Delta FK_\Delta^{-1} \quad (137)$$

So

$$g_\Delta(U_F) = U_S U_F^* U_S^{-1} \quad (138)$$

Applying this to the projector P_1 defined in Eq. (103) we deduce

$$g_\Delta(P_1) = U_S P_1^* U_S^{-1} \quad (139)$$

We have

$$\langle e_{j_1}^0|\psi\rangle\langle\psi|e_{j_2}^0\rangle = \lambda\langle e_{j_1}^0|P_1|e_{j_2}^0\rangle \quad (140)$$

for some constant λ . Consequently

$$g_\Delta(\langle e_{j_1}^0|\psi\rangle)g_\Delta(\langle\psi|e_{j_2}^0\rangle) = \frac{g_\Delta(\lambda)}{\lambda^*}(\langle e_{x^{-1}j_1}^0|\psi\rangle)^*(\langle\psi|e_{x^{-1}j_2}^0\rangle)^* \quad (141)$$

which is easily seen to imply

$$g_\Delta(\langle e_j^0|\psi\rangle) = \mu(\langle e_{x^{-1}j}^0|\psi\rangle)^* \quad (142)$$

for some constant μ . By repeated application of this formula we find

$$g_\Delta^l(\langle e_j^0|\psi\rangle) = \begin{cases} \mu_l(\langle e_{x^{-l}j}^0|\psi\rangle)^* & l \text{ odd} \\ \mu_l\langle e_{x^{-l}j}^0|\psi\rangle & l \text{ even} \end{cases} \quad (143)$$

where $\mu_l = g_{\Delta}^{l-1}(\mu)g_{\Delta}^{l-2}(\mu^*) \dots$. Hence

$$|\langle e_{p_l(j)}^{z_l} |\psi \rangle|^2 = |\mu_l|^2 |\langle e_{x^{-l}j}^0 |\psi \rangle|^2 \quad (144)$$

Since

$$\sum_j |\langle e_j^{z_l} |\psi \rangle|^2 = \sum_j |\langle e_j^0 |\psi \rangle|^2 = 1 \quad (145)$$

we must have $|\mu|^2 = 1$, from which it follows that the normalized state

$$|\phi \rangle = \frac{1}{\sqrt{\| |\psi \rangle \|}} |\psi \rangle \quad (146)$$

is MUB-balanced. \square

The MUB-balanced states whose existence is established by this theorem are identical with the ones constructed using a different method by Amburg *et al* [10]. In fact, the orbit of states constructed in ref. [10] is generated by the state with Wigner function

$$W_{\mathbf{p}} = \frac{1}{d(d+1)} \left(1 - d\delta_{\mathbf{p}, \mathbf{0}} + \sum_{x \in \mathbb{F}_d^*} l(x^2 + 1) \omega^{\text{tr}[x(p_1^2 + p_2^2)]} \right). \quad (147)$$

Define

$$F = \begin{pmatrix} \alpha & \beta \\ -\beta & \alpha \end{pmatrix}, \quad \alpha = \frac{1}{2}(\eta + \eta^d) \quad \beta = \frac{i_M}{2}(\eta - \eta^d) \quad (148)$$

where

$$i_M = \eta^{\frac{1}{4}(p-1)(d+1)} \quad (149)$$

(observe that $i_M^2 = -1$, so i_M is a modular analogue of i). It is easily seen that $\alpha^d = \alpha$ and $\beta^d = \beta$. So $F \in \text{GL}(2, \mathbb{F}_d)$. Moreover

$$\Delta = \det(F) = \eta^{d+1}, \quad t = \text{Tr}(F) = \eta + \eta^d. \quad (150)$$

So F is a MUB-cycler. The fact that

$$(\alpha p_1 + \beta p_2)^2 + (-\beta p_1 + \alpha p_2)^2 = \Delta(p_1^2 + p_2^2) \quad (151)$$

means $W_{F\mathbf{p}} = g_{\Delta}(W_{\mathbf{p}})$, which is easily seen to imply that the state corresponding to $W_{\mathbf{p}}$ is an eigenstate of U_F (*modulo* multiplication by a scalar).

It is interesting to ask how many MUB-balanced states there are. Consider the MUB-cycler U_{G_0} , where G_0 is the matrix defined by Eq. (90) with multiplicative order $2m_0 = p - 1$. Let $|\psi \rangle$ be the corresponding MUB-balanced state. Since $|\psi \rangle$ is the unique eigenstate of $U_{G_0}^{p-1}$ with eigenvalue 1 it will be left invariant by any Clifford unitary or anti-unitary V such that

$$V U_{G_0}^{p-1} V^{-1} = U_{G_0}^{t(p-1)} \quad (152)$$

for some t . One finds that the only possibilities are

$$V = U_{G_0^{u(p-1)/2}} \quad (153)$$

(corresponding to $t = 1$) or

$$V = U_{G_0^{u(p-1)/2} F}, \quad F = \begin{pmatrix} 0 & i_M \eta^{\frac{d+1}{2}} \\ i_M \eta^{-\frac{d+1}{2}} & 0 \end{pmatrix} \quad (154)$$

(corresponding to $t = -1$), where in both cases u can take any integral value in the range $0 \leq u < 2(d+1)$ (to understand the form of the matrix F observe that $FG_0F^{-1} = \eta^{d+1}G_0^{-1}$, implying $FG_0^{p-1}F^{-1} = G_0^{-(p-1)}$). We conjecture that there are no other extended Clifford unitaries or anti-unitaries leaving $|\psi\rangle$ invariant (we have checked this conjecture in detail for $d = 7, 11, 19$). Since the order of the extended Clifford group is $2d^3(d^2 - 1)$ (see ref. [40]) that would mean that the number of MUB-balanced states is $d^3(d-1)/2$.

We were surprised by the results obtained in this section, as we had expected that our construction would yield many new MUB-balanced states, additional to the ones described by Amburg *et al.* However, our results suggest that Amburg *et al.* have in fact constructed the entire set. If so it would mean that such states form a highly distinguished geometrical structure. Amburg *et al.* describe MUB-balanced states as “states which have no right to exist”. The same could be said of SICs. But whereas there are, in most of the dimensions which have been examined, several different orbits of SICs, it looks as though there is only one orbit of MUB-balanced states. If that were indeed the case it would mean that such states are very special indeed.

12. SUMMARY

We have had to go through a large amount of detailed arguments, and our main results are summarized in theorems 1, 2, 5, 6, and 12. At the same time the picture we have arrived at is simple and appealing.

First, the g-unitaries themselves. They generalize the notion of anti-unitaries, but their action is restricted to vectors taking values in some special number field. We have been concerned with what is arguably the simplest example, when the number field is generated by some root of unity. Then the g-unitaries do play a role in the description of complete sets of MUB in odd dimensions, and as far as the transformations of the actual MUB vectors themselves are concerned they have a simple interpretation as rotations in Bloch space—just as ordinary unitaries always have.

Mutually unbiased bases are interesting from many points of view. Here we have been interested in the projective transformations that permute them, and we saw that provided the dimension is a prime power $d = p^n$ where n is odd the g-unitaries provide us with transformations that cycle through the entire set of $d+1$ bases. If the dimension equals $3 \bmod 4$ some of these transformations are effected by anti-unitaries, but when the dimension equals $1 \bmod 4$ it is necessary to turn to g-unitaries for this purpose. We have shown that every MUB-cycling g-unitary leaves one vector in Hilbert space invariant, and that this eigenvector has definite parity. If $d = 3 \bmod 4$ this eigenvector is also an eigenvector of an anti-unitary operator which is itself MUB-cycling. (In even prime power dimensions unitary MUB-cyclers exist [7, 8].)

If $d = 3 \bmod 4$ the eigenvectors of MUB-cycling g-unitaries are MUB-balanced states in the sense of Wootters and Sussman [7, 12] and Amburg *et al.* [10]. Our construction is a useful supplement to the work of Amburg *et al.* for two reasons. In the first place it gives additional insight into the features of Hilbert space which are responsible for the existence of such states. In their original paper Wootters and Sussman showed that one gets MUB-balanced states in the even prime power case by taking eigenvectors of cycling unitaries. Amburg *et al.* demonstrated the

existence of such states in odd prime power dimension equal to $3 \bmod 4$. They also gave a very appealing formula for the Wigner function of such a state. However, they did not demonstrate the connection with the even prime power case. In this paper we have demonstrated such a connection: to go from the even prime power case to the odd prime power case with $d = 3 \bmod 4$ one merely has to replace the cycling unitaries of Wootters and Sussman with cycling g -unitaries. In the second place the fact that we have exposed the underlying structural reasons for the existence of MUB-balanced states suggests the conjecture that we have in fact found *every* such state. We had expected that our construction would yield many new MUB-balanced states, additional to those found by Amburg et al. However, our results suggest that such states are confined to a single orbit of the extended Clifford group. If that is the case it would mean that they are very remarkable states indeed, which we feel may well repay further investigation. In this connection we would particularly like to draw the reader's attention to the fact that the distribution of their components is governed by Wigner's semi-circle law when d is large [10] (a fact which has attracted the interest of the pure mathematics community [41]).

Finally, as we noted in the introduction, the concept of a g -unitary originally arose in connection with the SIC-problem [2]. Specifically the known SIC fiducials are all eigenvectors of a family of g -unitaries. In this paper we have proved a number of results connected with the problem of finding the eigenvectors of a g -unitary. Our discussion of this problem was not exhaustive, being confined to the case of MUB-cycling g -unitaries defined over a cyclotomic field. The problem of extending our analysis to the g -unitaries which arise in the SIC problem is non-trivial. Nevertheless, our results may be regarded as a useful first step in the direction of solving that more difficult problem.

ACKNOWLEDGEMENTS

IB thanks Jan-Åke Larsson for criticism at an early stage. DMA thanks Steve Flammia for illuminating discussions concerning the eigenvectors of g -unitaries. David Andersson provided useful information about MUB-balanced states. HBD was supported by the Natural Sciences and Engineering Research Council of Canada and the Vanier Canada Graduate Scholarship. DMA was supported by the IARPA MQCO program, by the ARC via EQuS project number CE11001013, and by the US Army Research Office grant numbers W911NF-14-1-0098 and W911NF-14-1-0103.

REFERENCES

- [1] E. Wigner: *Group Theory and its Application to the Quantum Mechanics of Atomic Spectra*, Academic Press, New York 1959.
- [2] D. M. Appleby, H. Yadsan-Appleby, and G. Zauner, *Galois automorphisms of a symmetric measurement*, Quantum Inf. Comp. **13** (2013) 672.
- [3] V. I. Arnold: *Dynamics, Statistics, and Projective Geometry of Galois Fields*, Cambridge UP 2011.
- [4] W. K. Wootters, *A Wigner-function formulation of finite-state quantum mechanics*, Ann. Phys. **176** (1976) 1.
- [5] I. Bengtsson and Å. Ericsson, *Mutually unbiased bases and the complementarity polytope*, Open. Sys. Information Dyn. **12** (2005) 107.
- [6] C. Cormick, E. F. Galvão, D. Gottesman, J. P. Paz, and A. O. Pittenger, *Classicality in discrete Wigner functions*, Phys. Rev. A **73** (2006) 012301.

- [7] W. K. Wootters and D. M. Sussman, *Discrete phase space and minimum-uncertainty states*, eprint quant-ph/0704.1277.
- [8] U. Seyfarth, L. L. Sanchez-Soto, and G. Leuchs, *Structure of the sets of mutually unbiased bases with cyclic symmetry*, eprint arXiv:1404.3035.
- [9] D. M. Appleby, *Properties of the extended Clifford group with applications to SIC-POVMs and MUBs*, arXiv:0909.5233.
- [10] I. Amburg, R. Sharma, D. Sussman and W.K. Wootters, *States that “look the same” with respect to every basis in a mutually unbiased set*, eprint arXiv:1407.4074.
- [11] D. M. Appleby, H. B. Dang, and C. A. Fuchs, *Symmetric Informationally-Complete quantum states as analogues to orthonormal bases and Minimum Uncertainty States*, Entropy **16** (2014), 1484.
- [12] D. M. Sussman: *Minimum-Uncertainty States and Rotational Invariance in Discrete Phase space*, BSc thesis, Williams College 2007.
- [13] N. Schappacher, *On the history of Hilbert’s 12th problem: A comedy of errors*, in *Matériaux pour l’histoire des mathématiques au XXth siècle*, Soc. Math. France, Paris 1998.
- [14] A. J. Scott, *Tight informationally complete quantum measurements*, J. Phys. A **39** (2006) 13507.
- [15] B.-G. Englert D. Kaszlikowski, H. K. Ng, W. K. Chua, J. Řeháček and J. Anders, *Efficient and robust quantum key distribution with minimal state tomography*, quant-ph/0412075.
- [16] B. Chen, T. Li and S. Fei, *General SIC-measurement based entanglement detection*, arXiv:1406.7820.
- [17] C. A. Fuchs and R. Schack, *Quantum-Bayesian Coherence: The no-nonsense version*, arXiv:1301.3274.
- [18] G. Zauner: *Quantendesigns. Grundzüge einer nichtkommutativen Designtheorie*, PhD thesis, Univ. Wien 1999. Available in English translation in Int. J. Quantum Inf. **9** (2011) 445.
- [19] J. M. Renes, R. Blume-Kohout, A. J. Scott, and C. M. Caves, *Symmetric informationally complete quantum measurements*, J. Math. Phys. **45** (2004) 2171.
- [20] D. M. Appleby, *SIC-POVMs and the extended Clifford group*, J. Math. Phys. **46** (2005) 052107.
- [21] A. J. Scott and M. Grassl, *Symmetric informationally complete positive-operator-valued measures: A new computer study*, J. Math. Phys. **51** (2010) 042203.
- [22] W. K. Wootters and B. D. Fields, *Optimal state-determination by mutually unbiased measurements*, Ann. Phys. **191** (1989) 363.
- [23] I. Stewart: *Galois Theory*, Chapman and Hall, London 1972.
- [24] J.S. Milne: *Fields and Galois Theory*, available online at www.jmilne.org/math/.
- [25] S. Roman: *Field Theory*, Graduate Texts in Mathematics no. 158, Springer, New York, 2005.
- [26] I. D. Ivanović, *Geometrical description of state determination*, J. Phys. **A14** (1981) 3241.
- [27] S. Bandyopadhyay, P. O. Boykin, V. Roychowdhury and F. Vatan, *A new proof for the existence of mutually unbiased bases*, Algorithmica **34** (2002) 512.
- [28] D. M. Appleby, I. Bengtsson, and S. Chaturvedi, *Spectra of phase point operators in odd prime dimensions and the extended Clifford group*, J. Math. Phys. **49** (2008) 012102.
- [29] M. Neuhauser, *An explicit construction of the metaplectic representation over a finite field*, J. Lie Theory **12** (2002) 15.
- [30] D. Gottesman and I. L. Chuang, *Demonstrating the viability of universal quantum computation using teleportation and single-qubit operations*, Nature **402** (1999) 390.
- [31] G.J. Janusz: *Algebraic Number Fields*, Graduate Studies in Mathematics vol. 7, American Mathematical Society, Providence RI, 1996.
- [32] A. Vourdas, *Quantum systems with finite Hilbert space: Galois fields in quantum mechanics*, J. Phys. **A40** (2007) R285.
- [33] D. Gross, *Hudson’s theorem for finite-dimensional quantum systems*, J. Math. Phys. **47** (2006) 122107.
- [34] M. K. Bennett: *Affine and Projective Geometry*, Wiley, New York 1995.
- [35] Å. Ericsson: *Exploring the Set of Quantum States*, PhD thesis, Univ. Stockholm 1999.
- [36] K. S. Gibbons, M. J. Hoffman, and W. K. Wootters, *Discrete phase space based on finite fields*, Phys. Rev. **A70** (2004) 062101.
- [37] A. Casaccino, E. F. Galvão, and S. Severini, *Extrema of discrete Wigner functions and applications*, Phys. Rev. **A78** (2008) 022310.
- [38] E. Wigner, *Normal form of antiunitary operators*, J. Math. Phys. **1** (1960) 409.

- [39] W. Greub: *Linear Algebra*, Graduate Texts in Mathematics no. 23, Springer, New York, 1975.
- [40] K. E. Gehles: *Ordinary Characters of Finite Special Linear Groups*, MSc Thesis, Univ. of St. Andrews 2002.
- [41] N. M. Katz, *Rigid local systems and a question of Wootters*, Commun. Number Theory and Physics **6** (2012) 223.